3.1 Seq., Series, common

\[ F_{n} \text{ converges uniformly } (\text{domain}) \]
\[ \exists > 0 \quad N \quad (m, n \geq N \text{ and } z \in R) \]
\[ |f_{m}(z) - f_{n}(z)| < \varepsilon \]
\[ z \in R \quad \exists f_{n}(z) \quad \text{point-seq.} \]

define \( g(z) = \lim_{n \to \infty} f_{n}(z) \)

\[ f_{n} \to g \text{ uniformly uniform limit.} \]

Thus, \( \exists f_{n} \text{ uniform limit } g \).

and for all \( n \), \( f_{n} \) is continuous,

then \( g \) is continuous and if

\[ C \text{ is a path in } R, \]
\[ \lim_{n \to \infty} \int_{C} f_{n} = \int_{C} g \]
\[ |\int_{C} f_{n} - \int_{C} g| \leq \int_{C} |f_{n} - g| \leq \varepsilon \cdot \lambda(C) \]
Thm 112. Weierstrass "M" test. Suppose \( \exists b_j \geq M_j \text{ for } j \in \mathbb{N} \) such that \( |f_j(z)| \leq b_j \). Then if \( \sum b_j \) converges, \( \sum f_j \) converges uniformly in \( R \). (Also converges absolutely in \( R \)).

Sec. 3.2 Taylor Series

\[
\sum_{n=0}^{\infty} A_n (z-a)^n
\]

\( S_0(w) = A_0 \)

\( S_1(w) = A_0 + A_1 (z-w) \)

\( S_2(w) = A_0 + A_1 (z-w) + A_2 (z-w)^2 \)

3 Thus about power series

1) \( w \) A S converges at \( z \neq w \)
1. $|x-w| < |z-w|$

2. $\sum_{n=1}^{\infty} a_n (z \in D)$ converges absolutely at $x$ and uniformly on each closed subset of $D$.

3. Exactly one of the following holds (for $S = \sum_{n=1}^{\infty} a_n$):
   a) $S$ is totally convergent
   b) $S$ is totally divergent
   c) $S$ has a radius of convergence $r$ such that if $|x-w| < r$, $S$ converges at $x$; if $|x-w| > r$, $S$ diverges at $x$.

3. $s_n = \left| a_n \right|^{1/m}$
   a) final set of $S$ is unbounded
   b) totally divergent
b) \( \lim_{n \to \infty} S_n = 0 \) totally convergent

c) \( \frac{1}{r} \) greatest cluster pt of \( s \)

Prof. Sec. 3.2

1. a) \( \sum_{n=1}^{\infty} \frac{z^n}{n!} \) What is \( A_n \) ? \( A_n = 1 \)

\( S_n = \frac{1}{A_n} = 1 \)

\( r = \) greatest CP \( r = 1 \)

b) \( \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \)

\( A_n = \frac{1}{(n+1)!} \)

\( S_n = \{ \frac{1}{(n+1)!} \} \)

\( (2n)! > n^n \)

\( (2n)! \cdot \frac{1}{2n} > n^n \cdot \frac{1}{2n} \)

\( \frac{1}{(2n)!} \cdot \frac{\frac{1}{2n}}{\frac{1}{2n}} > \frac{1}{n^n} \)

\( \frac{1}{(2n)!} \cdot \frac{\frac{1}{2n}}{\frac{1}{2n}} < \frac{1}{n^n} \)

So \( \lim_{n \to \infty} S_n = 0 \) \( \Rightarrow \) \( S \) totally convergent.
\[ \sum_{n=1}^{\infty} \frac{1}{(n+1)!} z^n = z \sum_{n=1}^{\infty} \frac{1}{(n+1)!} z^n = z^{-1} \sum_{n=1}^{\infty} \frac{z^n}{1+z^2} \cdot \sum_{n=1}^{\infty} \frac{z^n}{1+z^2} = e^z - (1+z) \quad \text{Totally Conv.} \]

3) \[ \sum_{n=1}^{\infty} n^2 z^n \quad A_n = n \quad \exists n = n \quad \text{Final set unbounded} \Rightarrow S \text{ tot. div.} \]

**Def.** If \( K \) is a closed path and \( z \notin K \), then the winding number of \( K \) about \( z \), denoted by \( W(K, z) \), is the integer \( \frac{1}{2\pi i} \int_{C} \frac{1}{z-z} \)

\[ \int_{C} \frac{1}{z-z} \, dz \]

\[ \int_{K} \frac{1}{z-z} \, dz \]