ODE II Feb 6, 2007 $\dot{x} = Ax$ $\dot{x} = f(x)$

Thm (Banach) page 212 - Arnold

Suppose $A: M \to M$ is a contraction mapping of a complete metric space into itself. Then A has a unique fixed-point β and if $x_0 \in M$, $\lim_{n\to\infty} A^n x_0 = \beta$.

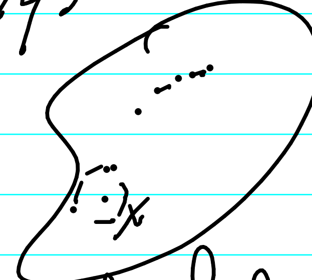
Pf M complete metric space

(M, ρ) $x, y \in M$

1) $\rho(x, y) \geq 0$ and $\rho(x,y) = 0$ iff $x = y$

2) $\rho(x,y) = \rho(y,x)$

3) $\rho(x,z) + \rho(z,y) \geq \rho(x,y)$



$\{x_n\}$ converges or is a Cauchy sequence if: for every $\varepsilon > 0$ there is a pos. int. N such that if n, m are pos. int. not less than N, then $\rho(x_n, x_m) < \varepsilon$.

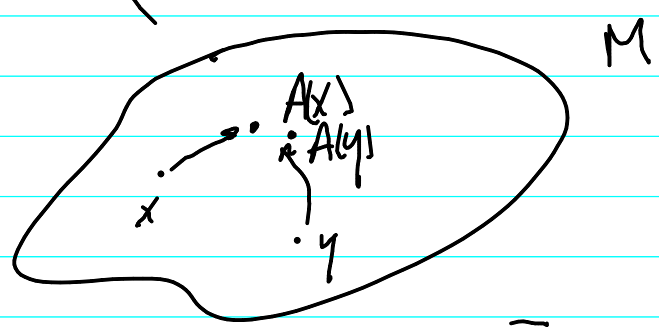
$\{x_n\}$ has limit y

$\varepsilon > 0 \quad N \quad n \geq N \quad \rho(x_n, y) < \varepsilon$

A metric space in which every Cauchy sequence has a limit is said to be <u>complete</u>.

$A: M \to M$ is a <u>contraction</u> mapping if there exists a K, $0 < K < 1$, such that for all x, y in M we have

$$\rho(A(x), A(y)) \leq K \rho(x, y)$$



Uniqueness - suppose $\beta, \bar{\beta}$ are fixed-points of A, i.e., $A(\beta) = \beta$ and $A(\bar{\beta}) = \bar{\beta}$.

$\rho(\beta, \bar{\beta}) = \rho(A(\beta), A(\bar{\beta})) \le \underbrace{K\, \rho(\beta, \bar{\beta})}$

Suppose $\rho(\beta, \bar{\beta}) \neq 0$. Then

$< 1 \cdot \rho(\beta, \bar{\beta})$

Then $\rho(\beta, \bar{\beta}) < \rho(\beta, \bar{\beta})$ $\times\!\!\!\times$ Contradiction

$\rho(\beta, \bar{\beta}) = 0 \implies \beta = \bar{\beta}$.

Scratch Work: Pick $x_0 \in M$

Build sequence $x_0, A(x_0), A(A(x_0)), \ldots$

$\{A^n(x_0)\}_{n=0}$. Suppose $\lim_{n \to \infty} A^n(x_0) = \beta$

exists. Argue that β must be a fixed-point

$$A\left(\lim_{n \to \infty} A^n(x_0)\right) = A(\beta)$$

(apply A to both sides).

If A can be shown to be continuous

$$\beta = \lim_{n \to \infty} A^{n+1}(x_0) = A(\beta)$$

What do we mean by "A is continuous at x"?

$\varepsilon > 0$ $\delta > 0$ s.t. if $y \in M$ & $\rho(x,y) < \delta$ then $\rho(A(x), A(y)) < \varepsilon$.

$\varepsilon > 0$

$\delta = \underline{\frac{\varepsilon}{K}}$

Let $y \in M$ $\rho(x,y) < \delta$

$\rho(A(x), A(y)) \le K\,\rho(x,y) < K \cdot \delta = \varepsilon$

So A is continuous. (Same argument for Lipshitz A.)

All that is left is to show that $\{A^n(x_0)\}$ is a Cauchy sequence.

Scratch work.

$\varepsilon > 0$

$N = \_\_\_\_\_\_\_\_\_\_$?

$n, m \ge N$

$$\rho(A^m(x_0), A^n(x_0)) < \varepsilon.$$

Convenient to use alternate def. of Cauchy seq.

$\varepsilon > 0$ there is a positive int. m such that if n is any positive int., $\rho(A^{m+n}(x_0), A^m(x_0)) < \varepsilon$

$$\rho(A^{m+n}(x_0), A^m(x_0)) \leq$$

$$K\,\rho(A^{m+n-1}(x_0), A^{m-1}(x_0)) \leq$$

$$K^2\,\rho(A^{m+n-2}(x_0), A^{m-2}(x_0)) \leq$$

$$\cdots$$

$$K^m\,\rho(A^n(x_0), x_0) \leq$$

$$K^m\Big\{\rho(A^n(x_0), A^{n-1}(x_0)) + \rho(A^{n-1}(x_0), A^{n-2}(x_0)) + \cdots + \rho(A^2(x_0), A(x_0)) + \rho(A(x_0), x_0)\Big\}$$

$$\le K^{m}\left\{ K^{n-1}\rho(A(x_0), x_0) + K^{n-2}\rho(A(x_0), x_0) + \cdots \right.$$

$$\left. \cdots + K\rho(A(x_0), x_0) + \rho(A(x_0), x_0) \right\}$$

$$\le K^{m}\rho(A(x_0), x_0)\left\{ \underbrace{K^{n-1} + K^{n-2} + \cdots + K + 1}_{\frac{K^{n}-1}{K-1}} \right\}$$

$$< \rho(A(x_0), x_0)\,\frac{K^{m}}{1-K}$$