Assignment #2  From Trefethen & Bau

Lecture 2  p. 15-16
2.1, 2.2, 2.3, 2.4, 2.6
Due Tues. Feb. 27

Discussion

2.1 \( Q^*Q = I \)

2.2
\[
1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}
\]

\[
\begin{array}{c}
4 \vdots + \\
3 \vdots + \\
\vdots \\
4 + 3 \vdots + \\
4 + 3 \vdots + \\
\vdots + \\
1 + 2 + 3 + \cdots + n = \frac{(n+1)n}{2}
\end{array}
\]

2.3 \( A^* = A \)

2.5 \( u^*u \) inner product \( u^*u = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \vdots \\
\end{bmatrix} \begin{bmatrix} u_1 \\
\vdots \\
u_n \end{bmatrix} \) real

\( uu^* \) outer product

\( E(u, v; \sigma) = I - \sigma uu^* \) Householder
SVD Proof

Let \( \sigma_i = \| A \|_2 \). By compactness there is a unit vector \( \tilde{v}_i \in \mathbb{C}^n \) such that \( A \tilde{v}_i = \sigma_i \tilde{v}_i \).

Extend \( \tilde{v}_i, \tilde{u}_i \) to an orthonormal basis \( \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n \) of \( \mathbb{C}^n \) and extend \( \tilde{u}_i, \tilde{v}_i \) to an orthonormal basis \( u_1, u_2, \ldots, u_n \) of \( \mathbb{C}^n \). Let \( U \) and \( V \) be the unitary matrices with columns \( u_j \) and \( v_j \) respectively, \( u_j^* u_i = \delta_{ij} \).

\[
U^* A V_i = S = \left[ \begin{array} { c c } { \tilde{\sigma}_1, } & { w^* } \\
{ B } & { } \end{array} \right] = \sigma_i^2
\]

where \( \tilde{\sigma} \) is a column vector of \( \dim \, m-1 \), \( w^* \) is a row vector of \( \dim \, n-1 \), and \( B \) is an \((m-1) \times (n-1)\) matrix.

\[
A V_i = u_i^* v_i, \quad \quad u_1^* u_i = \frac{\sigma_i}{\sigma_1}, \quad \quad u_i^* u_1 = \frac{\sigma_1}{\sigma_i} = \sigma_i
\]
How do we extend \( \xi V, \xi \) to an orthonormal basis?

\[ \xi V, \xi \] \[ \| V \| = 1 \]

\( W_2 \) linearly independent of \( V_1 \), we can produce a vector \( \xi \) to \( V_1 \) by projection

\[ r = W_2 - (V_1^*W_2)V_1 \]

\[ \text{Proj onto span } V, \]

\[ V_1^*r = V_1^*W_2 - V_1^* \{ (V_1^*W_2)V_1 \} \]

\[ = V_1^*W_2 - (V_1^*W_2)V_1V_1^* \]

\[ = 0 \]

\( r \) is orthogonal to \( V_1 \).

Define \( V_2 = \frac{r}{\| r \|} \).

\( \xi V_1, V_2 \) an orthonormal set

that spans the same space as \( \xi V_1, W_2 \).

Can be repeated, sort of by the name of Gram-Schmidt.

\( \xi W_1, W_2, \ldots, W_k \) linearly independent

it produces an orthonormal set

\( \xi V_1, V_2, \ldots, V_k \) which spans the same
\[ G = W_3 - \left\{ (V_i^* W_3) V_i + (V_z^* W_3) V_z \right\} \]

\[ = (I - P) W_3 \]

Consider \( S = \begin{bmatrix} \sigma_i & w^* \\ \overline{\sigma} & B \end{bmatrix} \)

\[ \| \begin{bmatrix} \sigma_i & w^* \\ \overline{\sigma} & B \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \|_2 \geq \sigma_i^2 + w^* w \]

\[ = (\sigma_i^2 + w^* w)^{1/2} \| [\sigma_i] \|_2 \]

\[ \therefore \| S \|_2 \geq (\sigma_i^2 + w^* w)^{1/2} \]