Num. Linear Alg. Feb 20

Seminar Professor Helton UCSD

Friday 3:00 PM BSE 2102

Assign #2

2.1 If $A$ is unitary $\Rightarrow$ diagonal

Take $3 \times 3$ example

$A^*A = I \quad A^* = A^{-1}$

2.2 $(AB)_{ij} = \sum_k A_{ik} B_{kj}$

$\| x_1 + x_2 \|^2 = \| x_1 \|^2 + \| x_2 \|^2$

$\langle \sum_{i=1}^n x_i, \sum_{j=1}^n x_j \rangle$

2.3 $A^*A = A \quad A \vec{e}_i = \lambda \vec{e}_i \quad \vec{e}_i \perp \vec{e}_j$

$\lambda = \lambda$

(0) $A \vec{e}_1 = \lambda_1 \vec{e}_1 \quad A \vec{e}_2 = \lambda_2 \vec{e}_2$
For $x$ in $\mathbb{C}^n$, $f(x) = \langle Ax, x \rangle$ is called a quadratic form.

2.4 eigenvalues of unitary $Q$.

$$Q^* Q = I \quad \lambda \lambda = 1 \quad |\lambda|^2 = 1$$

$$\begin{align*}
(a-bi)(a+bi) \\
a^2 + b^2 = |\lambda|^2
\end{align*}$$

2.6 $A = I + u v^*$

$$A^{-1} = I + \alpha u v^*$$

$$\begin{bmatrix}
u_1 & \vdots & \mu_1 \\
u_2 & \vdots & \mu_2 \\
u_3 & \vdots & \mu_3
\end{bmatrix}$$

SVD: $\sigma_i = |\lambda_i|$. Since $|\lambda_i| = \sup \|Av\|$ on the unit sphere in $\mathbb{C}^n$ is compact, there is a $v_i \in \mathbb{C}^n$ such that $\|v_i\| = 1$ and $\|Av_i\| = \sigma_i$. Let $u_i = Av_i$.

Extend $\xi v_i \perp \xi$ basis of $\mathbb{C}^n$.

Extend $\xi u_i = \frac{v_i}{\|v_i\|}$ to basis of $\mathbb{C}^n$. 

\[ u v^* u v^* \]
Let $U, V$ be unitary matrices with columnwise $\Sigma_i x_i y_i$ and $\Sigma_i y_i z_i$, respectively.

$U^* A V = U^* [A_{ij} = \mu] V$ \[= \begin{bmatrix} \sigma_i & w^* \\ 0 & B \end{bmatrix} \approx S \]

$\mathbf{w}^* \mathbf{u} = \frac{\mathbf{u}^* \mathbf{w}}{||\mathbf{u}||} = \frac{||\mathbf{u}||}{||\mathbf{u}||} = 1$ \(\sigma_i^2 \), where $\sigma_i$ is an $m \times 1$ column vector, $w^*$ is a row vector of length $(n-1)$ and $B$ is $(m-1) \times (n-1)$ matrix.

$\| \begin{bmatrix} \sigma_i & w^* \\ 0 & B \end{bmatrix} \|_2 \geq \sigma_i^2 + w^* w$ \[\approx (\sigma_i^2 + w^* w)^{1/2} \|w\|_2 \]

$|S|_2 \geq \sqrt{\sigma_i^2 + w^* w}$. However,

since $U, V$ are unitary,

$|S| = |A| = \sigma_i$, so \(\mathbf{w} = \mathbf{0}\).
If \( n = 1 \) or \( m = 1 \), done. Otherwise, let \( B \) describe action of \( A \) on \( \perp \) complement to \( \text{span} \{ u_1, u_2 \} \). By induction argument \( B \) has SVD

\[
B = U_2 \Sigma_2 V_2^*
\]

Then

\[
A = U_1 \begin{bmatrix} \mathbf{1} & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} \mathbf{0} & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ 0 & V_2 \end{bmatrix}^*
\]

is a SVD of \( A \).

"Sec. 5" More on SVD

Change of basis: SVD \( \Rightarrow \) every linear transformation has a diagonal matrix representation in orthogonal proper bases in domain and range spaces.

\[
b = Ax \iff u^*b = u^*Ax \iff u^*b = \frac{u^*u}{2} \Sigma V^* x
\]
Defining $b' = \mathbf{U}^* b$ and $x' = \mathbf{V}^* x$, then $b = \mathbf{Ax}$ iff $b' = \sum x'$.

A reduced Σ- diagonal for Σ when range space is expressed in the basis of left-singular vectors (columns of $\mathbf{U}$ since $\mathbf{U} \mathbf{b} = b$) and domain space $\mathbb{R}^n$ is expressed in basis of right singular vectors (cols. of $\mathbf{V}$).

App. V Image Processing

Satellite picture, 1000 x 1000 pixels, color scale in range between black and white and send each 10^6 numbers.

Instead find essential information and send only that.

Perform an SVD; if we find that only the first 60 are significant singular values
- above a threshold we keep them, and below we throw them away.
- The remaining 940, and send only the corresponding 60 columns of II and V.