Case 3(b) One linear eigenvector, geometric multiplicity 1

Ex. \[ \begin{align*}
  \dot{x} &= -2x \\
  \dot{y} &= x - 2y \\
  \begin{bmatrix}
    -2 & 0 \\
    1 & -2
  \end{bmatrix}
\end{align*} \]

Solve \[ x(t) = Ae^{-2t}, \quad y(t) = Be^{-2t} + Ate^{-2t} \]

Phase portrait more complicated: suppose \( A > 0 \), then \( x > 0 \) and \( e^{-2t} = \frac{x}{A} \),

\[ t = -\frac{1}{2} \ln \left( \frac{x}{A} \right), \]

\[ y = \frac{Bx}{A} - \frac{x}{2} \ln \left( \frac{x}{A} \right) \]

All trajectories approach the origin as \( t \to \infty \). Slope at any point,

\[ \begin{align*}
  \frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\
  \frac{dy}{dx} &= \frac{-2Be^{-2t} + Ate^{-2t} - 2Ae^{-2t}}{-2Ae^{-2t}} \\
  &= \frac{-2B + A}{-2A} \to \infty \text{ as } t \to \infty
\end{align*} \]

Trajectories enter the origin along the
y-axis. Again the CP is an improper node.

Case 4 Complex Roots \( \lambda = \alpha + \beta i \):
\[
\begin{align*}
(x(t)) &= c_1 e^{\alpha t} \cos(\beta t) \Re \vec{e} + \\
(y(t)) &= c_2 e^{\alpha t} \sin(\beta t) \Im \vec{e}
\end{align*}
\]
\( \alpha < 0 \) motion toward CP  
\( \alpha > 0 \) motion away CP

Ex: \( x = -x + 2y \)  
\( y = -2x - y \)
\[
\begin{align*}
x(t) &= e^{-t} (A \cos 2t + B \sin 2t) \\
y(t) &= e^{-t} (B \cos 2t - A \sin 2t)
\end{align*}
\]
Let \( x = r \cos \theta, y = r \sin \theta, \)
\( R = (A^2 + B^2)^{\frac{1}{2}}, \quad R \cos \phi = A, \)
\( R \sin \phi = B, \) we have
\[
\begin{align*}
r \cos \theta &= R e^{-t} \cos (2t - \phi) \\
r \sin \theta &= -R e^{-t} \sin (2t - \phi)
\end{align*}
\]
So that \( r = R e^{-t}, \theta = -(2t - \phi), \)
and eliminating \( t, \) gives
\[
r = R e^{-\frac{\theta - \phi}{2}}
\]
giving a family of spirals. \( \alpha = -1 < 0 \)
spirals toward \( CP, \) \( \theta \) decreases
with increasing \( t, \) motion clockwise.

SPIRATIONAL POINT
Case 5: Pure Imaginary roots. Motion is periodic in time and trajectories are closed curves.

**CENTER**

\[
\begin{array}{c|c}
\lambda &= -(a+d)\lambda +(ad-bc)=0 \\
\text{ad} - bc &\neq 0
\end{array}
\]

**Table**

<table>
<thead>
<tr>
<th>Roots</th>
<th>CP</th>
<th>Stability</th>
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</thead>
<tbody>
<tr>
<td>(\lambda_1 &gt; \lambda_2 &gt; 0)</td>
<td>Improper</td>
<td>Unstable</td>
</tr>
<tr>
<td>(\lambda_1 &lt; \lambda_2 &lt; 0)</td>
<td>Node</td>
<td>Asymptotically stable</td>
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<tr>
<td>(\lambda_2 &lt; 0 &lt; \lambda_1)</td>
<td>Saddle Point</td>
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<tr>
<td>(\lambda_1 = \lambda_2 &gt; 0)</td>
<td>Proper or</td>
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</tr>
<tr>
<td>(\lambda_1 = \lambda_2 &lt; 0)</td>
<td>Improper Node</td>
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<tr>
<td>(\lambda = \alpha \pm \beta i)</td>
<td>Spiral Pt</td>
<td>Unstable</td>
</tr>
<tr>
<td>(\alpha &gt; 0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\alpha &lt; 0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ \lambda_1 = \beta i, \quad \lambda_2 = -\beta i \quad \text{Center Stable} \]

**Lorenz System** \[ u = f(u) \]

\[ \begin{cases} 
  \dot{x} = -\sigma x + \sigma y \\
  \dot{y} = -xy - yz + \epsilon \\
  \dot{z} = -bz + xy + b(\tau + \sigma)
\end{cases} \]

Here \( b, \tau, \sigma > 0 \) constants

Derived from a version of the Navier-Stokes equations used for weather prediction after many (stastic) simplifying assumptions.

James Gleick's Chaos

The variation equations for the system are \( \dot{u} = A(u)u \) where \( u \) is a sol. to \((*)\) and

\[ A(u) = \begin{bmatrix} -\sigma & 0 & 0 \\
  -\sigma - z & -1 & -x \\
  y & x & -b \end{bmatrix} \]
Since \( \text{Tr} A(u) = -\alpha -1 - b < 0 \), the region occupied by the time-asymptotic trajectories, the attractor, has dimension less than 3.

Construction Middle Third Cantor Set

\[
\begin{align*}
S_0 &: 0 \quad \frac{\ln 2}{\ln 3} \quad 1 \\
S_1 &: 0 \quad \frac{1}{3} \quad \frac{\ln 3}{\ln 3} \quad 1 \\
S_2 &: 0 \quad \frac{\ln 3}{9} \quad \frac{1}{3} \quad \frac{3}{9} \quad 1 \\
S_n &: \quad \bigcap S_n = K
\end{align*}
\]