Then \( f \) has the point function if it is analytic in \( R \) and \( w \in R \):

1. If \( f \) does not have negative order at \( w \), then \( f(z) = 0 \) for \( z \) in \( R \).
2. If \( f \) does not have order at \( w \), then \( f(z) = 0 \) for \( z \) in \( R \).
3. If \( f \) has order \( j \) at \( w \), then \( j \) must be an integer and there is a function \( g \), analytic in \( R \), such that \( g(w) \neq 0 \), and

\[
f(z) = (z - w)^j g(z)
\]

for \( z \) in \( R \) different from \( w \).

**Proof:** Let \( f \), \( R \), \( w \),... 
1. Let \( r > 0 \) such that \( S = \overline{D_r(w)} \subset R \).
2. Let \( z \) in initial set of \( f \) and

\[
0 < |z - w| < r.
\]
2) Suppose the set of numbers $v$ with property (x) is not bounded above. Let $k$ be a nonnegative integer and $\varepsilon > 0$. Then $|f(x)| \leq |z_0 - w|^\alpha (1 + |f_z + 1|)$.

Let $n > n + 2$ with prop. (x) for $r > 0$, $M > 0$.

Let $p > 0$ such that $D(w) < R$ and $\varepsilon < \min \{\varepsilon, \frac{\varepsilon}{M^{n+1}}, \frac{\varepsilon}{2}\}$.

$$|f(w)| = \left| \frac{n!}{2\pi i} \int \frac{f}{(1 - w)^{n+1}} \right|$$

$$K = c_p(w)$$

$$\leq \frac{n!}{2\pi} \frac{1 + K'}{p^{n+1}} \cdot \frac{2\pi p}{2\pi p}$$

$$\leq n! \frac{1}{p^{n+1}} \rho \cdot M \frac{\varepsilon}{M^{n+1}} < \rho \cdot 3 < 3$$
So \( f^{(n)}(w) = 0 \) for all \( n \geq 0 \) and \( f \) is the constant 0 function.

(3) \( f \) has order \( l \) at \( w \)

Let \( r > 0 \) s.t. \( D_r(w) \subset \mathbb{R} \)

Let \( l \) be least non-negative integer \( n \) such that \( f^{(n)}(w) \neq 0 \).

On \( D_r(w) \)

\[
\begin{align*}
\sum_{p=0}^{\infty} \frac{f^{(p+1)}(w)}{(p+1)!} (I-w)^{p+1} &= \sum_{p=0}^{\infty} \frac{f^{(p+1)}(w)}{(p+1)!} (I-w)^{p+1} \\
&= (I-w)^l \sum_{p=0}^{\infty} \frac{f^{(p+1)}(w)}{(p+1)!} (I-w)^{p+1} \\
&= (I-w)^l \sum_{p=0}^{\infty} \frac{f^{(p+1)}(w)}{(p+1)!} (I-w)^{p+1}
\end{align*}
\]

Define \( g \) on \( \mathbb{R} \) by

\[
\begin{cases}
\sum_{p=0}^{\infty} \frac{f^{(p+1)}(w)}{(p+1)!} (z-w)^{p+1} & z \in \overline{D_r(w)} \\
\frac{f(z)}{(z-w)^l} & z \in \mathbb{R} - \overline{D_r(w)}
\end{cases}
\]
\( g(w) \neq 0, f(z) = (z-w)^l \cdot g(z) \) for \( z \neq w \),
and \( l \) has property (\(*\)).

Suppose \( v > l \) and \( v \) has (\(*\)) with
\( \epsilon > 0 \) and
\[ \rho = \min\{ \frac{\epsilon}{2}, \left( \frac{e}{l!M} \right)^{\frac{1}{v-l}} \} \]
\[ |f'(w)| \leq \frac{l!}{2\pi i} \int_{|z-w|=\rho} \frac{f(z)}{(z-w)^{l+1}} dz \]
\[ (e^{\rho}) \leq \frac{l!}{2\pi i} \frac{\rho \cdot M}{\rho^{l+1} \cdot 2\pi e} \]
\[ = e^{\rho-l} \cdot l! \cdot M \leq \epsilon \quad \text{**Contradiction} \]

Hence \( l = j \) Order of \( f \) at \( w \).

Thus 41: Suppose \( g \) is an entire function,
\( w \) is a point and each of \( \nu, R, M \)
is a pos. number such that if \( z \)
it a point and \( |z-w| > R \), then
\[ |g(z)| \leq |z-w|^\nu \cdot M \text{ if } n \]
is an integer greater than $v$, then $g^{(n)}(w) = 0$, so that either $\xi$ is constant or $\xi$ is a polynomial of degree not greater than $v$.

**Proof.** Assume $\xi$ is a polynomial of degree $v$. Let $n$ be an integer greater than $v$, and let $\varepsilon > 0$.

Let $\xi > R$ s.t. $L(r) > \frac{1}{(v-n)} L \left( \frac{\varepsilon}{n+1} \right)$.

$$\left| g^{(n)}(w) \right| = \left| \frac{n!}{2\pi i} \int \frac{\xi}{(1-w)^{n+1}} \right| \underbrace{C_{r}(w)}_{\text{For } z \in C_{r}(w),}

\leq \frac{n!}{2\pi} \frac{1}{|1-w|^{n+1}} \left| C_{r}(w) \right| \cdot 2\pi R \cdot 2\pi R

\frac{1}{|z-w|^n} \leq \frac{1}{R^{n+1}}, \quad \text{so}

\left| g^{(n)}(w) \right| \leq n! M R^{-n} \underbrace{< n! M \frac{\varepsilon}{n+1}}_{m = n+1} = \varepsilon
\( Since \ (v-n) \ L(t) < L \left( \frac{\varepsilon}{n! M} \right) \epsilon \)

\[ \Rightarrow \ L \left( \frac{v-n}{t} \right) < L \left( \frac{\varepsilon}{n! M} \right). \]