\[ e^A = \lim_{n \to \infty} (I + \frac{1}{n} A)^n = \lim_{n \to \infty} (I - \frac{1}{n} A)^{-n} \]

\[ \sin z = \sum_{k=1}^{\infty} \frac{z^{2k}}{k!} \quad \text{for a new sequence } \sum_{n=1}^{\infty} \frac{a_n}{n!} \text{ where} \]

\[ p_n = a_1 a_2 \ldots a_n = \prod_{k=1}^{n} a_k \]

the \( n \)th partial product of \( \sum_{n=1}^{\infty} \frac{a_n}{n!} \)

\[ a_i a_2 \ldots \frac{a_n}{n!} \]

\( a_k \) is the \( k \)th factor or \( k \)th term if the product. Could say product is convergent if \( p_n \to 0 \) in analogy to series. Then every even product having one factor equals \( 0 \) would be convergent. While deletion of these factors...
could bring about divergence.

0.2.3

Def. must be changed.

First suppose \( a_k \neq 0 \) for all \( k \).

If \( P_n - P \in \mathbb{R} - \{0\} \), infinite product is convergent to \( P \). General case product is convergent if

1) There exists an \( N > 0 \) such that for \( n > N \), \( a_n \neq 0 \).

2) \( a_{N+1}, a_{N+2}, \ldots \) is convergent to say \( \alpha \).

Value of the product is \( P = a_1 \cdot a_2 \cdot a_3 \ldots \).

Value of a convergent if. product is 0

iff at least one of its factors is 0

Ex. \( \prod_{n=1}^{\infty} (1 + \frac{1}{n}) \) divergent \( \forall 0 + \infty \)

\[
\prod_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \ldots \cdot \frac{n+1}{n}}{n+1} = n+1
\]
Ex. \( \prod_{n=2}^{\infty} \left( 1 - \frac{1}{n} \right) \) divergent \( \Rightarrow 0 \)

\( P_n = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{n} = \frac{1}{n} \) partial products tend \( \Rightarrow 0 \).

Ex. 0.1.1.1 \cdots \) convergent \( \Rightarrow 0 \)

A necessary and sufficient condition for convergence of \( \prod_{k=1}^{\infty} a_k \) is that for every \( \varepsilon > 0 \) there is an \( N > 0 \) such that if \( n > N \) and \( k > 0 \),

\[ |a_{n+1} \cdot a_{n+2} \cdots a_{n+k} - 1| < \varepsilon. \]

(Analogue of Cauchy's criterion).

\[ \Rightarrow \text{let } \varepsilon > 0 \text{ and } \delta = \min \{ \varepsilon \frac{1}{4}, \frac{1}{\varepsilon} \}, \frac{1}{\delta} \leq \delta. \]

\[ N > 0 \text{ and } n > N \text{ and } k > 0, \]

\[ \left| \prod_{j=1}^{n+k} a_j - 1 \right| < \delta \]

\[ \text{let } n > N \text{ and } k > 0, \]

\[ \left| \prod_{j=1}^{n+k} a_j - 1 \right| < \delta \leq \frac{3}{4} \]
\[
\frac{1}{n} \prod_{j=1}^{n} a_j - \varepsilon < s \leq \frac{3|\Theta|}{4}
\]

\[
\left| \frac{1}{n} \prod_{j=1}^{n} a_j - \varepsilon \right| < \frac{3|\Theta|}{2}
\]

\[
\left| \frac{1}{n} \prod_{j=1}^{n} a_j - 1 \right| < \frac{3|\Theta|}{2}
\]

\[
\frac{1}{n} \prod_{j=1}^{n} a_j - 1 \left| \leq \frac{1}{n} \frac{3|\Theta|}{2}
\]

\[
\leq \frac{2}{|\Theta|} \frac{3|\Theta|}{2} = 3
\]

Since \[
|\Theta| - \frac{1}{n} \prod_{j=1}^{n} a_j \leq |\Theta| - \frac{1}{n} \prod_{j=1}^{n} a_j \leq \frac{|\Theta|}{2}
\]

\[
\frac{|\Theta|}{2} \leq \frac{1}{n} \prod_{j=1}^{n} a_j \leq \frac{2}{|\Theta|} \geq \frac{1}{n} \prod_{j=1}^{n} a_j
\]

\[
\leq \text{let } 0 < \varepsilon < 1
\]

\[
\text{let } N_1 > 0 \text{ if } n > N_1, \text{ if } k > 0, \text{ then}
\]
\[ 1 \prod_{j=n+1}^{u+k} a_j - 1 \mid < \varepsilon. \]

If \( n > N_1 + 1 \), then \( a_n \neq 0 \). Moreover,\[ \prod_{j=N_1+1}^{N_1+k} a_j \mid < 1 + \varepsilon \quad \text{for every } \varepsilon. \]

Let \( N_1 \). Then \[ \prod_{j=1}^{N_1} \prod_{j=N_1+1}^{N_1+k} a_j \mid \leq (1 + \varepsilon) \prod_{j=1}^{N_1} \bigg| a_j \bigg| \]

Let \( R = (1 + \varepsilon) \prod_{j=1}^{N_1} \bigg| a_j \bigg| \)

Let \( N_2 > 0 \) if \( n > N_2 \) and \( k > 0 \). \[ \prod_{j=n+1}^{n+k} a_j \mid < \frac{\varepsilon}{2 (1 + \varepsilon)} \]

Let \( N = N_2 \), \( n > N \) and \( k > 0 \)

\[ \mid p_{n+k} - p_n \mid = \prod_{j=1}^{n+k} a_j - \prod_{j=1}^{n} a_j \mid \]
\[ \prod_{i=1}^{n} a_i \prod_{i=n+1}^{n+k} a_i - 1 \]
\[ = \prod_{i=1}^{n} a_i \prod_{i=n+1}^{n+k} a_i - 1 \]
\[ \leq \prod_{i=1}^{n} \sqrt{a_i} \prod_{i=n+1}^{n+k} \sqrt{a_i} - 1 \]
\[ \leq \prod_{i=1}^{n} \sqrt{a_i} \prod_{i=n+1}^{n+k} \sqrt{a_i} - 1 \]
\[ \leq 3 < 3 \]

The seq. \( \{ \sum_{n=1}^{\infty} \} \) is Cauchy, hence has a limit \( L \) which is \( 0 \) only if \( a_j = 0, \ 1 \leq j \leq N+1 \).

Case 1: \( K = 1 \), rewrite cond. for convergence of \( \sum_{n=1}^{\infty} \).

Thus \( \sum_{n=1}^{\infty} \) converges to rewrite the product as \( \prod_{n=1}^{\infty} 1 + u_n \).

Necess. cond. for conv. \( \lim_{n \to \infty} u_n = 0 \).

Thus \( \sum_{n=1}^{\infty} u_n \) is a numberseq.

and for all \( n \), \( u_n \geq 0 \), then
Lemma. For all $x \in \mathbb{R}$, $1 + x \leq e^x$.

Clear for $x \geq 0$. Let $f(x) = e^x - (1 + x)$ for $x \leq 0$, then $f(x) = e^x - 1 \leq 0$ and $f$ is non-increasing on $(-\infty, 0]$.

$f(x) = f(0) - \frac{x}{1 + x} = 0$.

Proof: Let $k$ be a positive int. $x$.

$S_k = 1 + \sum_{n=1}^{k} u_n \leq P = \prod_{n=1}^{k} (1 + u_n) \leq \prod_{n=1}^{k} e = e^k$.

Seq. of partial products $\prod_{k \geq 1} S_k$ and seq. of partial sums $\sum_{k \geq 1} S_k$ are both non-decreasing hence have a limit if the have a bounded final set.

Let $\prod_{n=1}^{\infty} (1 + u_n)$ is absolutely convergent.
If \( \sum_{n=1}^{\infty} (1 + u_n) \) is convergent, then if \( \sum_{n=1}^{\infty} |u_n| \) is convergent. Otherwise, conditionally convergent.

Then if \( \prod_{n=1}^{\infty} (1 + u_n) \) is absolutely conv.

1) it is convergence in ordinary sense.
2) remains conv. after change of order of the factors.
3) value of product is unaltered by rearrangement of factors.

Consider functional products of form

\[
\prod_{n=1}^{\infty} (1 + u_n(x))
\]

where for all \( n \), \( u_n(x) \) is a point function. If product converges put \( z = \exp \sum \) for a set \( S \), and the seq.
If partial products \( \prod_{k=1}^{n} \mathfrak{P}_k(z) \) are uniformly convergent on \( S \).

Then if \( \sum_{n=1}^{\infty} \mathfrak{U}_n(z) \) is a seq. of analytic functions each with initial set \( R \) and the series \( \sum_{n=1}^{\infty} |\mathfrak{U}_n(z)| \) is uniformly convergent on \( R \). Then the product \( \prod_{n=1}^{\infty} \mathfrak{U}_n(z) \) is absolutely and uniformly convergent on \( R \) to an analytic function \( \mathfrak{F}(z) \) defined on \( R \) and \( \mathfrak{F}(z) = 0 \) for some \( z \) iff one factor in the product is zero.