Num. Lin. Al. March 8

Gram-Schmidt Given lin. indep. vectors $V = \{v_1, v_2, \dots, v_K\}$ in $\mathbb{R}^n$, construct an orthonormal set of vectors $Q = \{q_1, q_2, \dots, q_K\}$ such that span Q = span V.

(1) Normalize v_1. Define $q_1 = \frac{v_1}{\|v_1\|}$

(2) Project onto orth. complement of span $\{q_1\}$ = span $\{v_1\}$.

a) What is the proj. onto span $\{q_1\}$

$$Pv_2 = \langle q_1, v_2 \rangle q_1$$

b) $\perp$ proj $(I-P)v_2$ Define

$$w_2 = v_2 - \langle q_1, v_2 \rangle q_1$$

Define $q_2 = \frac{w_2}{\|w_2\|}$

Note span $\{v_1, v_2\}$ = span $\{q_1, q_2\}$

3) $$W_3 = V_3 - \sum_{j=1}^{2} \langle q_j, V_3 \rangle q_j$$

$$q_3 = \frac{W_3}{\|W_3\|}$$

Consider solving $Ax = b$ using a process of "relaxation" (in the older literature.

$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Make a guess at a solution, call it $\vec{x}^{old} = (x_1^0, x_2^0)$. Take the 1st equation and solve it for a "new x_1^N" assuming x_2^0 is exactly right.

$$4x_1^N - 1x_2^0 = b_1$$

$$x_1^N = (b_1 + 1x_2^0)/4$$

Play same game with 2nd equation.

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$$-1x_1^0 + 4x_2^N = b_2$$

$$x_2^N = (b_2 + 1x_1^0)4$$

New guess for our solution $x^N = (x_1^N, x_2^N)$. Now overwrite x^0 with x^N and perform the iteration again. Hopefully the sequence of iterates will converge to the solution of $Ax = b$.

Consider the general case where we wish to build a sequence of approx. solutions $x^0, x^1, x^2, \ldots, x^n, \ldots$

$$A_{11}x_1^{n+1} + A_{12}x_2^n + \cdots + A_{1m}x_m^n = b_1$$

$$A_{21}x_1^n + A_{22}x_2^{n+1} + \cdots + A_{2m}x_m^n = b_2$$

$$\vdots$$

$$A_{m1}x_1^n + A_{m2}x_2^n + \cdots + A_{mm}x_m^{n+1} = b_m$$

Solve each equation "simultaneously"

$$x_1^{n+1} = \frac{1}{A_{11}}\left(b_1 - A_{12}x_2^n - \cdots - A_{1m}x_m^n\right)$$

$$x_2^{n+1} = \frac{1}{A_{22}}\left(b_2 - A_{21}x_1^n - A_{23}x_3^n \cdots - A_{2m}x_m^n\right)$$

$$\vdots$$

Rewrite for analysis: 1st equation

$$x_1^{n+1} = \frac{1}{A_{11}}\left(A_{11}x_1^n + b_1 - A_{11}x_1^n - A_{12}x_2^n - \cdots - A_{1m}x_m^n\right)$$

$$x_2^{n+1} = \cdots$$

$$\vdots$$

A $\cancel{x_{ij}}$ A_{ij}

$$\vec{x}^{n+1} = \vec{x}^n + D^{-1}\left(\vec{b} - A\vec{x}^n\right)$$

Defines an iteration, given an initial guess $\vec{x}^0$

$$\boxed{\begin{array}{l}\text{Newton}\\ z_{n+1} = z_n - F'(z_n)^{-1}F(z_n)\end{array}}$$

Function G, ξ is a fixed-point of G if $G(\xi) = \xi$

$$Ax = b$$
$$0 = b - Ax$$
$$0 = D^{-1}(b - Ax)$$
$$x = x + D^{-1}(b - Ax) \equiv G(x)$$

x solves $Ax = b$ iff $G(x) = x$, i.e., x is a fixed-point of G.

x^0, $G(x^0)$, $G(G(x^0))$, ...