Num. Lin. Al.
March 8

Gram-Schmidt: Given lin. indep.
vector $V = \{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{R}^n$.
Construct an orthonormal set of
vector $Q = \{q_1, q_2, \ldots, q_k\}$ such that
span $Q = \text{span } V$.

1) Normalize $V_1$. Define $q_1 = \frac{v_1}{\|v_1\|}$

2) Project onto orth. complement
of $\text{span } Q_1$. Define $q_2$

a) What is the proj. onto $\text{span } Q_1$?

$P_{v_2} = \langle q_1, v_2 \rangle q_1$

b) $\perp$ proj $(I - P)v_2$ Define

$w_2 = v_2 - \langle q_1, v_2 \rangle q_1$

Define $q_2 = \frac{w_2}{\|w_2\|}$

Note $\text{span } Q_1, Q_2 \subseteq \text{span } Q_1, q_2$.
3) \[ W_3 = v_3 - \sum_{j=1}^{2} <g_j, v_3> g_j \]

\[ q_3 = \frac{W_3}{\lVert W_3 \rVert} \]

Consider solving \( Ax = b \) using a process of "relaxation" (as in the older literature).

\[
\begin{bmatrix}
4 & -1 \\
-1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

Make a guess at a solution, call it \( \overline{x}^{0 \text{old}} = (x^{0 \text{old}}_1, x^{0 \text{old}}_2) \). Take the 1st equation and solve it for a "new \( x_1^{\text{new}} \)" assuming \( x_2 \) is correct. Let exactly right.

\[ 4(x_1^{\text{new}} - 1) + x_2^{\text{new}} = b_1 \]

\[ x_1^{\text{new}} = \frac{(b_1 + 1x_2)}{4} \]

Play same game with 2nd equation.
\[-1x_1^0 + 4x_2^N = b_2\]
\[x_2^N = (b_2 + 1x_1^0)^4\]

New guess for our solution \(x^N = (x_1^N, x_2^N)\). Now overwrite \(x^0\) with \(x^N\) and perform the iteration again. Hopefully, the sequence of iterates will converge to the solution of \(Ax = b\).

Consider the general case where we wish to build a sequence of approximate solutions \(x^0, x_1, x_2, \ldots, x_n, \ldots\):

\[A_1x_1^{n+1} + A_{12}x_2^n + \cdots + A_{1m}x_m^n = b_1\]
\[A_{21}x_1^n + A_{22}x_2^{n+1} + \cdots + A_{2m}x_m^n = b_2\]
\[\vdots\]
\[A_{m1}x_1^n + A_{m2}x_2^n + \cdots + A_{mm}x_m^{n+1} = b_m\]
Solve each equation "simultaneously":

\[ x_1^{n+1} = \frac{1}{A_{11}} \left( b_1 - A_{12} x_2^n - \cdots - A_{1m} x_m^n \right) \]

\[ x_2^{n+1} = \frac{1}{A_{22}} \left( b_2 - A_{21} x_1^n - \cdots - A_{2m} x_m^n \right) \]

Rewrite for analysis: let equation

\[ x_1^{n+1} = \frac{1}{A_{11}} \left( A_{11} x_1^n + b_1 - A_{12} x_2^n - \cdots - A_{1m} x_m^n \right) \]

\[ x_2^{n+1} = \cdots \]

\[ \dot{x}^{n+1} = \dot{x}^n + D^{-1} (b - A \dot{x}^n) \]

defines an iteration, given an initial guess \( \dot{x}^0 \) [Newton]

\[ z_{n+1} = z_n - F(z_n, F(z_n)) \]

Function \( G, \varepsilon \) is a fixed point of \( G \) if \( G(\varepsilon) = \varepsilon \).
\[ A x = b \]
\[ 0 = b - A x \]
\[ 0 = D'(b - A x) \]
\[ x = x + D'(b - A x) \equiv G(x) \]

x solves \( Ax = b \) iff \( G(x) = x \), i.e., \( x \) is a fixed point of \( G \).

\( x^0, G(x^0), G(G(x^0)), \ldots \)