Complex II  March 27

Next Homework

Section 3.3  #1, 2, 3, 4

Residue Theorem  Thm 39 (8)

\[ f(z) = \sum_{p=0}^{n} \frac{g_p}{z - w_p} + g_{n+1}(z) \]

\[ \Rightarrow \frac{1}{2\pi i} \int f = \sum_{K} \text{Res}(f, w_p) W(K, w_p) \]

1. \( K = c_i (0) \)  \( n \in \mathbb{Z} \)  \( f_n(z) = z^n \)

\[ \frac{1}{2\pi i} \int f_n = \begin{cases} 
1 & \text{if } n = -1 \\
0 & \text{otherwise} 
\end{cases} \]

2. Evaluate \( \frac{1}{2\pi i} \int z e^{\frac{1}{z}} \, dz = \frac{1}{z} \)

\[ z = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \]

\[ = z + 1 + \frac{z}{2} + \frac{z}{3} + \frac{z}{4} + \cdots \]

Residue at 0 is \( \frac{1}{2} \)
2. Winding number about \( \Omega \) is 1

\[ \frac{1}{z-w_1} + \ldots + \frac{1}{c_{m-2} - w_n} \]

3. Evaluate \( \oint_c \frac{z+2}{z(z+1)} \, dz \)

Simple pole at \( z = -1 \)
Use partial fractions to determine the residue.

\[ \frac{z+2}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1} \]

\[ z+2 = A(z+1)+Bz \quad z = -1 \quad B = -1 \]

\[ \oint_c \frac{2}{z} + \frac{-1}{z+1} \, dz \]

\[ \text{Res} (f, 0) = 2 \quad \text{Res} (f, -1) = -1 \]

\[ = 2\pi i \left( \frac{2}{0} - 1 \right) = 2\pi i \]

(Thm 38) \( q_0 (z) = 2z \), \( q_1 (z) = -(z+1) \), \( h(z) = 0 \)

\[ q_0 (-1 + \frac{1}{z+1}) = \frac{-1}{z+1} \]
3. Recall \( b_1 = \lim_{z \to w} \frac{1}{(K-1)!} \frac{d^{K-1}}{dz^{K-1}} \left( \frac{z-w}{z-w} \right)^{\frac{1}{2}} \).

If \( w \) is a pole of order \( K \).

4. Evaluate \( \frac{1}{2\pi i} \oint_{C(w)} \frac{3z+1}{z(z-1)^3} \, dz \).

- **Simple pole, pole order 3**

\( \text{Res}(f, 0) K=1 \lim_{z \to 0} \frac{3z+1}{z(z-1)} = -1 \)

\( \text{Res}(f, 1) K=3 \lim_{z \to 1} 2! \frac{d}{dz} \left\{ \frac{3z+1}{z} \right\} \)

\[ = \frac{1}{2} \lim_{z \to 1} \frac{d}{dz} (3z + z^{-1}) = \frac{1}{2} \frac{d}{dz} \{ -z^{-2} \} = -2z^3 \]

\[ = -1 \]

\( b_1 \) is the residue of \( f \) at \( w \).

That complex number such that the restriction of \( f(z) \), \( \frac{K}{z-w} \), annihilates the derivative of \( f(z) \).
1) uniform convergence $f_1 + f_2 + f_3 + \ldots$

$$\frac{1}{2} = \frac{z}{z+2} + 1 = \frac{z}{z+2} + \frac{2}{z+2} + \frac{z}{z+2} + \ldots$$

2) Exercise: $0$ is an essential singularity of $\exp(-\frac{1}{z^2})$, but $\exp(-\frac{1}{z^2})$ together with the restriction of $\exp(-\frac{1}{z^2})$ to the set of all numbers different from 0 is "infinitely differentiable."