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How do we actually compute SVD?

SVD related to eigenvalue-eigenvector factorization of a symmetric matrix:

$$A = Q \Lambda Q^T \text{ where } QQ^T = I.$$

SVD **Thm** Any $m \times n$ matrix A can be factored

$$A = Q_1 \Sigma Q_2^T \quad \text{(orthogonal)(diagonal)(orth.)}$$

Columns of Q_1 ($m \times m$) are eigenvectors of AA^T, columns of Q_2 ($n \times n$) are eigenvectors of A^TA. The r singular values on the diagonal of Σ ($m \times n$) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.

For SPD matrices this is identical to $Q \Lambda Q^T$

For symmetric indefinite matrices, neg.

eigenvalues in Λ become positive in Σ, and $Q_1 \neq Q_2$.

For complex matrices, Σ remains real but Q_1 and Q_2 become unitary

$$A = U_1 \Sigma U_2^H$$

Columns of Q_1 and Q_2 in SVD give orthonormal bases for all four fundamental subspaces:

first r columns of Q_1 – column space of A
last $m-r$ columns of Q_1 – left nullspace of A
first r columns of Q_2 – row space of A
last $n-r$ columns of Q_2 – nullspace of A

SVD chooses those bases in a special way, more than just orthonormal. If A multiplies a column of Q_2, it produces a multiple of a column of Q_1

$$A Q_2 = Q_1 \Sigma$$

$$AA^T = (Q_1 \Sigma Q_2^T)(Q_2 \Sigma^T Q_1^T) = Q_1 \Sigma \Sigma^T Q_1^T$$

$$A^T A = (Q_2 \Sigma^T Q_1^T)(Q_1 \Sigma Q_2^T) = Q_2 \Sigma^T \Sigma Q_2^T$$

Q_1 is the eigenvector matrix for AA^T; the eigenvalue matrix is $\Sigma \Sigma^T$ $(m \times m)$ with $\sigma_1^2, \dots, \sigma_r^2$ on the diagonal.

Q_2 is the eigenvector matrix for $A^T A$: diagonal matrix $\Sigma^T \Sigma$ has the same $\sigma_1^2, \dots, \sigma_r^2$, but is $n \times n$.

$A = QR$ factorization Q unitary

QR-Algorithm R upper triangular

for finding complete set of eigenvalues, eigenvectors.

$A = QR$, $B = RQ$, $B = Q_1 R_1$

$\{v_1, v_2, v_3\}$ lin. indep. G.S

gen. $\perp$ $\{q_1, q_2, q_3\}$

$$q_1 = \frac{v_1}{\|v_1\| = R_{11}}$$

$$q_2 = \frac{v_2 - (q_1^* v_2) q_1}{\|v_2 - (q_1^* v_2) q_1\| = R_{22}}$$

$$q_3 = \frac{v_3 - (q_1^* v_3) q_1 - (q_2^* v_3) q_2}{\|v_3 - (q_1^* v_3) q_1 - (q_2^* v_3) q_2\| = R_{33}}$$

$$R_{11} q_1 = v_1$$

$$\underbrace{(q_1^* v_2)}_{R_{12}} q_1 + R_{22} q_2 = v_2$$

$$\underbrace{(q_1^* v_3)}_{R_{13}} q_1 + \underbrace{(q_2^* v_3)}_{R_{23}} q_2 + R_{33} q_3 = v_3$$

1. $v_1 = R_{11} q_1$
2. $v_2 = R_{12} q_1 + R_{22} q_2$
3. $v_3 = R_{13} q_1 + R_{23} q_2 + R_{33} q_3$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix}.$$

Used GS to get a QR factorization of the matrix $A = [v_1\ v_2\ v_3]$.

$A = QR$ Solve $Ax = b$

$$QRx = b$$

$Rx = Q^* Q R x = Q^* b$ Back-substitution