1 ODE II March 29

Finish up competing-species

Linearize about the critical point (X_e, Y_e)

$$F'\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1\cdot(\epsilon_1-\sigma_1 x-\alpha_1 y)+x(-\sigma_1) & -\alpha_1 x\\ -\alpha_2 y & 1\cdot(\epsilon_2-\sigma_2 y-\alpha_2 x)+(-\sigma_2 y)\end{bmatrix}$$

$$\begin{matrix}\frac{d}{dt}(x-X_e)\\ \frac{d}{dt}(y-Y_e)\end{matrix}\approx\begin{bmatrix}(\epsilon_1-\sigma_1 X_e-\alpha_1 Y_e)-\sigma_1 X_e & -\alpha_1 X_e\\ -\alpha_2 Y_e & [(\epsilon_2-\sigma_2 Y_e-\alpha_2 X_e)-\sigma_2 Y_e]\end{bmatrix}\begin{bmatrix}x-X_e\\ y-Y_e\end{bmatrix}$$

Use this to determine the type and stability of any critical point.

coexistence – $X_e\neq 0$, $Y_e\neq 0$ only possible in cases (c) and (d).

$$X_e=\frac{\epsilon_1\sigma_2-\epsilon_2\alpha_1}{\sigma_1\sigma_2-\alpha_1\alpha_2}\qquad Y_e=\frac{\epsilon_2\sigma_1-\epsilon_1\alpha_2}{\sigma_1\sigma_2-\alpha_1\alpha_2}$$

Since $\epsilon_1-\sigma_1 X_e-\alpha_1 Y_e=0=\epsilon_2-\sigma_2 Y_e-\alpha_2 X_e$

linearized equations reduce to

$$\frac{du}{dt} = -\sigma_1 \bar{X}_e u - \alpha_1 \bar{X}_e v$$

$$\frac{dv}{dt} = -\alpha_2 \bar{Y}_e u - \sigma_2 \bar{Y}_e v$$

eigenvalues:

$$\frac{-(\sigma_1 \bar{X}_e + \sigma_2 \bar{Y}_e) \pm \sqrt{(\sigma_1 \bar{X}_e + \sigma_2 \bar{Y}_e)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)\bar{X}_e \bar{Y}_e}}{2}$$

If $\sigma_1\sigma_2 - \alpha_1\alpha_2 < 0$, radicand pos. and greater than $(\sigma_1 \bar{X}_e + \sigma_2 \bar{Y}_e)^2$. 2 roots, one > 0, one < 0. Unstable saddle point.

If $\sigma_1\sigma_2 - \alpha_1\alpha_2 > 0$, radicand is less than $(\sigma_1 \bar{X}_e + \sigma_2 \bar{Y}_e)^2$. Roots: 2 real neg. or complex with neg. real parts

Roots cannot be complex, so CP asymptotically stable improper node.

Coexistence possible if $\sigma_1\sigma_2 > \alpha_1\alpha_2$.

(Diffusion-Reaction PDE's $u(x,y,t)$, $v(x,y,t)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + au + bv + cuv \\ \quad \| \qquad\quad \| \\ u_t = D_1 \Delta u \\ \dfrac{\partial v}{\partial t} = D_2 \Delta v + dv + eu + fuv \end{cases}$$
)

Liapounov's 2nd Method

Stability of CP of almost linear system usually determined by linearized system. However no conclusion can be drawn when CP is center of linear system. (undamped pendulum, predator-prey

Need to study "region of asymptotic stability" where trajectories starting within that domain approach CP. Need global information. Liapounov's second (direct) method generalizes physical principles for conservative systems.

(1) a rest position is stable iff the potential energy is a local minimum

(2) Total energy is a constant during any motion.

Undamped Pendulum - conservative mech. sys.

$$\theta''(t) + \frac{g}{\ell} \sin \theta(t) = 0$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{\ell} \sin x \end{cases} \quad \text{where } x = \theta(t),\ y = \theta'(t)$$

Potential energy U is work done in lifting pendulum above its lowest position

$$U(x,y) = mg\ell(1 - \cos x)$$

CPs of system $x = n\pi,\ y = 0 \quad n \in \mathbb{Z}$

Expect $x = 2n\pi,\ y = 0 \quad n \in \mathbb{Z}$ (pend. bob is vertical with weight down) to be

<u>stable</u>; $x = (2n+1)\pi$, $y = 0$, $n \in \mathbb{Z}$ (bob vertical, weight up) <u>unstable</u>.

Agrees with the fact that: at former CPs U is a minimum $= 0$, at latter points U is a max $= 2mgl$.

Total energy is + of potential + kinetic

$$V(x,y) = mgl(1 - \cos x) + \frac{1}{2} m l^2 y^2$$

Consider how V changes as one moves along a trajectory $(x(t), y(t))$.

"Total derivative"

$$\frac{d}{dt} V(x(t), y(t)) = V_x(x(t), y(t)) \frac{dx}{dt} + V_y(x(t), y(t)) \frac{dy}{dt}$$