Autonomous system \[
\begin{align*}
\dot{x} &= F(x, y) \\
\dot{y} &= G(x, y)
\end{align*}
\]
\(V(x, y) : D \to \mathbb{R}\) where \(D\) is a domain containing the origin. \(V\) is positive definite on \(D\) if \(V(0, 0) = 0\) and \(V(x, y) > 0\) for all other points in \(D\). (negative semi-definite \(\leq 0\), neg. semi-definite \(= 0\), pos. semi-definite \(> 0\))

Ex. \(V(x, y) = \sin(x^2 + y^2)\) pos. def. on \(x^2 + y^2 < \frac{\pi^2}{2}\)

\[V(x, y) = \frac{(x+y)^2}{2} \quad \text{pos. semi-definite.}\]

\[f(x) = ax^2 + bx + c\]

\[f(-x) = f(x) = 0 \quad \text{on} \quad y = -x\]

\[\phi \text{ convex} \quad x, y \quad t \in [0, 1]\]

\[\phi((1-t)x + ty) \leq (1-t) \phi(x) + t \phi(y)\]
Define \( V(x, y) = V_x(x, y)F(x, y) + V_y(x, y)G(x, y) \)
\[ = V_x(\phi(t), \psi(t)) \frac{d\phi(t)}{dt} + V_y(\phi(t), \psi(t)) \frac{d\psi(t)}{dt} \]
\[ = \frac{d}{dt} V(\phi(t), \psi(t)) \]

where \((\phi(t), \psi(t))\) is a trajectory that passes through the point \((x, y)\) at time \(t\).

Thus, suppose the origin is an isolated critical point of the autonomous system. There exists a function \( V \) that is \( C^1 \) and positive definite for which \( V \) is negative definite for a domain \( D \) containing \((0, 0)\). Then the origin is an asymptotically stable critical point. If \( V \) is negative...
semi definite, $(0,0)$ is a stable CP.

**Argument.** Assume $V(x,y) \leq 0$.

Let $c \geq 0$; consider level set (curve) given by $V(x,y) = c$. If $c = 0$, curve reduces to a single point $x = 0, y = 0$.

If $c > 0$ and sufficiently small, continuity of $V \Rightarrow$ closed curve containing the origin.

As $c \to 0$ closed curves shrink to the origin; moreover, a trajectory starting inside of a closed curve $V(x,y) = c$ cannot cross to outside.
So origin is a stable critical point. Recall
\[ \nabla V(x, y) = (V_x(x, y), V_y(x, y)) \]
is orthogonal to the curve \( V(x, y) = C \)
and here is points away from origin.
\[ V(x(s), y(s)) = C \]
\[ W \frac{dx}{ds} + V \frac{dy}{ds} = 0 \]
Consider angle of intersection of trajectory \((\phi(t), \psi(t))\) and the
double curve \( V(x, y) = C \) at the point \((x_1, y_1) = (\phi(t_1), \psi(t_1))\).
\[ V(x_1, y_1) = \nabla V(x_1, y_1) \cdot \left( \frac{d\phi}{dt}(t_1), \frac{d\psi}{dt}(t_1) \right) \]
where the second vector in the dot
product is the tangent to the traj.
RHS is product of magnitudes times
cosine of angle between them.

$\nabla (x, y) \leq 0 \iff \cos \theta = 0 \quad \text{or} \quad 90^\circ \leq \theta \leq 270^\circ$

Direction of motion on traj. is
inward or at worst tangent to the
curve. Trajectories starting inside
a closed curve $\nabla (x, y) = 0$ cannot
escape, so origin is a stable
CP. If $\nabla (x, y) < 0$, traj. passing
through points on curve deal
actually pointing inward, so
asymptotically stable CP.