1 Complex II  April 10

Fourier Transform for functions of a single variable.

Fourier Trans. from Fourier Series

Let $f$ be a function defined on $(-L, L)$ where $L>0$. Its Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where $$c_n = \frac{1}{2L}\int_{-L}^{L} f(y) e^{-\frac{in\pi y}{L}}\, dy$$

(here recall $\langle f, g\rangle = \int f \bar{g}$ ).

$$g_n(x) = e^{in\pi \frac{x}{L}} \quad n \in \mathbb{Z}$$

Proj. Thm $\quad f = \sum_n \langle g_n, f\rangle g_n$

$$\cos\left(\frac{n\pi x}{L}\right) + i \sin\left(\frac{n\pi x}{L}\right)$$

Try letting $L \to +\infty$ Writing

$K = \frac{n\pi}{L}$ gives

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left\{ \int_{-L}^{L} f(y) e^{-iKy} dy \right\} e^{iKx} \frac{\pi}{L}$$

As $L \to \infty$, the points $K$ get close together, and in the limit $K$ "becomes" a continuous variable; the sum becomes an integral, and $\Delta K = \frac{\pi}{L}$ becomes $dK$, giving

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[ \int_{-\infty}^{\infty} f(y) e^{-iKy} dy \right]}_{F(K)} e^{iKx} dK$$

so this can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(K) e^{iKx} dK$$

$F$ is called the Fourier transform of $f(x)$, also written as $\hat{f}$ and $\mathcal{F}(f)$

# Fourier Transform Pairs

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x s}\, dx$$

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Aside $\quad (Af)(x) = \int_a^b K(x,y) f(y)\, dy \quad x \in [a,b]$

$$K : [a,b] \times [a,b] \to \mathbb{R} \text{ or } \mathbb{C}$$

A operator or transform called an "integral transform" with kernel $K(x,y)$. $\quad f \to Af$

p. 53 Trefethen

$$f(\cdot) \to \sum_{j=0}^{n-1} q_j(\cdot) \int_{-1}^{1} \overline{q_j(x)}\, f(x)\, dx$$

$1, x, x^2, x^3, \dots, x^n \quad$ lin. indep.

GS $\Rightarrow$ Legendre polynomials

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Rewrite using change-of-variables

$$F(s) = C\int_{-\infty}^{\infty} f(x)\, e^{-ixs}\, dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{ixs}\, ds$$

or in symmetric form.

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ixs}\, dx$$

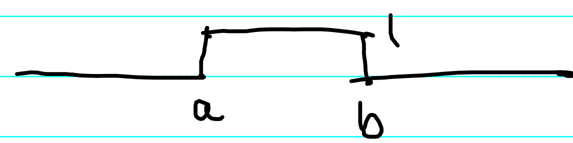
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{ixs}\, ds$$

In the last "pair" $\mathcal{F}$ has the property that $\mathcal{F}^{*} = \mathcal{F}^{-1}$, i.e., $\mathcal{F}$ is unitary: $\mathcal{F}^{*}\mathcal{F} = I$

If $f(x)$ and $F(s)$ are a transform pair in 1, then $f(x)$ and $F\left(\frac{s}{2\pi}\right)$ are a transform pair in 2, and $f\left(\frac{x}{\sqrt{2\pi}}\right)$ and $F\left(\frac{s}{\sqrt{2\pi}}\right)$ are a trans. pair in 3.

## Fourier Transform Table

1. Square pulse



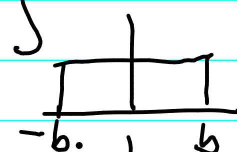
$$f(x) = \begin{cases} 0 & x < a \\ 1 & a \le x \le b \\ 0 & x > b \end{cases}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_a^b e^{-i\omega x}\, dx$$

$$= -\frac{1}{i\omega} e^{-i\omega x} \Big|_{x=a}^{b} \qquad \omega \neq 0$$

$$= \frac{1}{-i\omega} \left\{ e^{-i\omega b} - e^{-i\omega a} \right\}$$

2. Square Pulse centered at 0 


$$\hat{f}(\omega) = \frac{e^{-i\omega b} - e^{i\omega b}}{-i\omega} = 2\,\frac{\sin b\omega}{\omega}$$

$$= 2b\,\mathrm{sinc}(b\omega)$$


What is 2) as $b \to \infty$?

$$\hat{f}_b \longrightarrow 2\pi\,\delta(\omega)$$

Area of Δ is $2\pi$