Num. linear Algebra  April 10
Topics we need to cover: MGS-OR, Householder reflections, S-least squares, standard iterative methods, Jacobi & G-Seidel, conjugate gradient. Time permitting - hybrid subspace method, multigrid, FFT & wavelet.
See G. Strang, Bull. AMS April 1993 "Wavelet Transform vs. Fourier Transform."
Lecture 8: Gram-Schmidt Orth.
$A \in \mathbb{C}^{m \times n}$, $m \geq n$, be a full-rank matrix with columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$.
The GS iteration can be expressed as $q_i = \frac{1}{\|P_i a_i\|} P_i a_i$, $i = 1, \ldots, n$ where $P_i \in \mathbb{C}^{m \times m}$ is the orthogonal
projectors of rank $m-(j-1)$ that projects from onto the orthogonal complement to span $\mathfrak{E}g_{j}, \ldots, g_{j-1}$. (Here $P_j = I_m$)

g_j \perp g_{j+1}, \ldots, g_{j-1} \text{ and lies in span } \mathfrak{E}a_{j}, \ldots, a_{j-3}$.

To represent $P_j$ explicitly define $\hat{Q}_{j-1}$ to be an $m \times (j-1)$ matrix

$$\hat{Q}_{j-1} = \begin{bmatrix} g_1 & g_2 & \cdots & g_{j-1} \end{bmatrix}$$

Then $P_j = I - \hat{Q}_{j-1} \hat{Q}_{j-1}^*$

Modified GS - Original GS algorithm is numerically unstable and should be modified. Note

$P_j = P_1g_{j-1} \cdots P_1g_2 P_1g_1$
where \( P_{\perp q} \) is the rank \( m-1 \) orth. projector onto \( \text{span}(\mathbf{e}_q^3)^\perp \). MGS evaluates in order

\[
\begin{align*}
v_j &= \mathbf{a}_j, \\
v_j &= P_{\perp \mathbf{q}}, \quad v_j = v_j - \mathbf{e}_j \mathbf{e}_j \cdot v_j, \\
v_j &= v_j - P_{-\mathbf{q}}v_j, \quad v_j = v_j - \mathbf{e}_j \mathbf{e}_j \cdot v_j.
\end{align*}
\]

2nd way to visualize equivalence of (and difference in) MGS vs. GS

Let \( Q_j \) denote \( j \) proj. onto span(\mathbf{e}_q^3) so \( I = P_j + Q_j \). Then

\[
\begin{align*}
P_j &= P_{\perp \mathbf{q}_j} \cdots P_{\perp \mathbf{q}_1} \\
&= (I - Q_{j-1}) \cdots (I - Q_1) \quad \text{MGS} \\
&= I - (Q_1 + Q_2 + \cdots + Q_j) \quad \text{GS}
\end{align*}
\]
“cross-terms” of the form $Q_j Q_k$ etc. are zero in finite-precision arithmetic but can lead to instability when roundoff error is included in analysis.

Implementation: $P_j$ can be applied to $V_i$ for each $j > i$ once $V_i$ is known.

\[
\begin{align*}
\text{for } i &= 1 \to n \\
V_i &= a_{i,i} \\
\text{for } i &= 1 \to n \\
V_i &= \|V_i\|_1 \\
g_i &= V_i / \|V_i\|_1 \\
\text{for } i &= i+1 \to n \\
V_i &= g_i * V_i \\
V_i &= V_i - \bar{V_i} \cdot g_i
\end{align*}
\]
Operation Count. GS in either form requires \( n^2 \) \( 2mn^2 \) operations (flops) to compute a QR factorization of an \( m \times n \) matrix. For large work dominated by innermost loop: 

\[
\sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{4m}{i} \approx \frac{4m}{2} \frac{n(n+1)}{2} = 2mn^2
\]

\( m=n \) op. cost in \( \Theta(2n^3) \) so singular to LU fac. but with a larger constant. \( \checkmark \)
GS Triangular Orthogonalization

Each output step of MGS is right-multiplication by a square upper-D triangle matrix

\[
\begin{bmatrix}
\ell_{11} & \ell_{12} & \ell_{13} & \cdots \\
0 & \ell_{22} & \ell_{23} & \cdots \\
0 & 0 & \ell_{33} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_1 & V_2 & \cdots & V_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_1^{(2)} & V_2^{(2)} & \cdots & V_n^{(2)}
\end{bmatrix}
\]

At the end of the iterations we have

\[
A R_1 R_2 \cdots R_n = \hat{Q} \hat{R}^{-1} \quad \hat{A} = \hat{Q} \hat{R}
\]

So MGS method of $A \perp \perp$. 