1 ODE II. April 10

Periodic Solutions & Limit Cycles

Periodic sols. of nonlinear autonomous systems - trajectories form closed curves in phase plane. (Equilibrium sol. $x(t)=x_0$, $y(t)=y_0$ corresponding to $CP=(x_0,y_0)$ is special case of periodic sol., with any period)

For linear autonomous system

$$\dot{x} = ax+by$$
$$\dot{y} = cx+dy$$

periodic sol. iff roots of

$$\lambda^2 - (a+d)\lambda + (ad-bc)=0$$

are pure imaginary (in fact all sols. are periodic).

Nonlinear auto. system very different.

$$\dot{x} = x + y - x(x^2+y^2)$$
$$\dot{y} = -x + y - y(x^2+y^2)$$

Only CP is (0,0); system almost linear in nhbd of origin, which is unstable spiral point. Might think all trajectories spiral out to infinity.

Introduce polar coordinates r, θ

$$x = r\cos\theta \qquad y = r\sin\theta$$

Now multiply first eq. by x, second by y and add

$$x\dot{x} + y\dot{y} = x^2 + \cancel{xy} - x^2(x^2+y^2) + \cancel{-xy} + y^2 - y^2(x^2+y^2)$$
$$= (x^2+y^2) - (x^2+y^2)^2$$

$r^2 = x^2+y^2 \qquad r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$, so

$$r\frac{dr}{dt} = r^2(1-r^2)$$

If $r>1$, $\frac{dr}{dt}<0$ direction of motion on Trajectory is <u>inward</u>

If $r<1$, $\frac{dr}{dt}>0$ direction of motion <u>outward</u>

Circle $r=1$ special significance.

Equation for θ. Multiply first eq. by y, second by x, and subtract

$$y\dot{x} - x\dot{y} = \cancel{yx} + y^2 - \cancel{xy(x^2+y^2)}$$
$$x^2 - \cancel{xy} + \cancel{xy(x^2+y^2)}$$
$$= x^2+y^2$$

LHS is $r\sin\theta\{\dot{r}\cos\theta - r\sin\theta\,\dot{\theta}\}$

$$-\,r\cos\theta\{\dot{r}\sin\theta + r\cos\theta\,\dot{\theta}\}$$
$$= -r^2\dot{\theta}\{\sin^2\theta + \cos^2\theta\}$$
$$= -r^2\frac{d\theta}{dt}$$

Above reduces to $\frac{d\theta}{dt} = -1$

One solution of this system for r

4

and $\dot{\Theta}$ is

$$r(t) = 1, \quad \Theta(t) = -t + t_0$$

where t_0 is an arbitrary constant.

Other sols. found by separation of variables: if $r \neq 0$, $r \neq 1$

$$\frac{1}{r(1-r^2)} \frac{dr}{dt} = 1$$

Can use partial fractions to obtain

$$r(t) = \frac{1}{\sqrt{1 + c_0 e^{-2t}}}, \quad \Theta(t) = -t + t_0$$

c_0, t_0 constants.

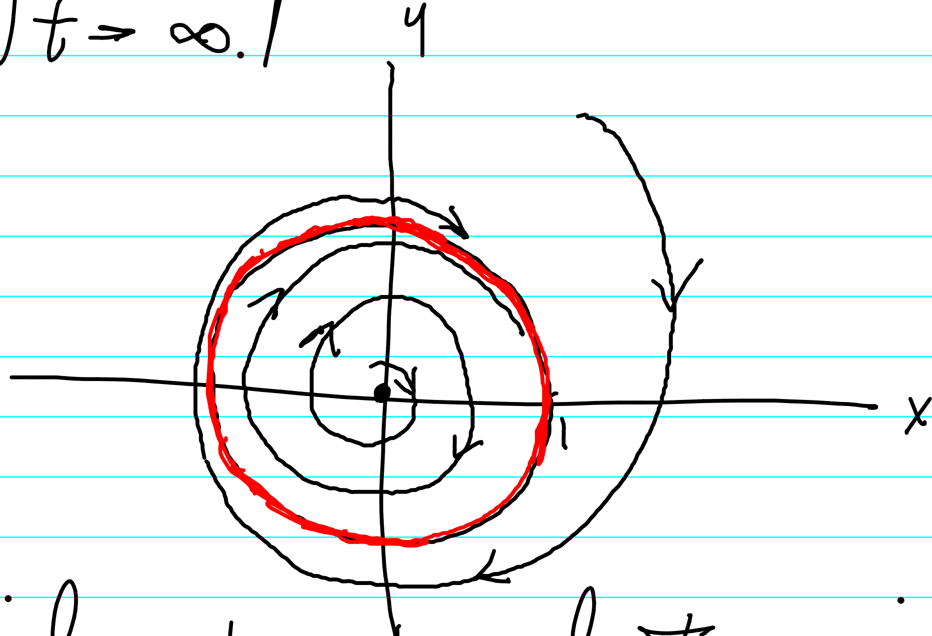
Solution satisfying initial conditions $r = \rho$, $\Theta = \alpha$ at $t = 0$ is given by

$$r(t) = \frac{1}{\sqrt{1 + [(1/\rho^2) - 1] e^{-2t}}}, \quad \Theta(t) = -(t - \alpha)$$

$0 < \rho < 1$, $r(t) \to 1$ from inside as $t \to \infty$

$\rho > 1$, $r(t) \to 1$ from outside as $t \to \infty$

Trajectories spiral toward circle $r=1$ as $t \to \infty$.



Circle $r=1$ corresponds to a periodic solution, which in this case is also a limit cycle.

A closed curve in phase plane which has nonclosed curves spiraling toward it from the inside or outside as $t \to \infty$ is called a limit cycle.

If all traj. spiral toward the LC,

it is stable. If Traj. on one side spiral toward LC and on the other spiral away, it is semistable. If Traj. on both sides recede, as $t \to \infty$ the LC is unstable.
(linear periodic case neutrally stable)