Periodic Solutions & Limit Cycles

Periodic solns. of nonlinear autonomous systems - trajectories form closed curves in phase plane. (Equilibrium soln. \( x(t) = x_0 \), \( y(t) = y_0 \) corresponding to \( CP = (x_0, y_0) \) is special case of periodic soln. with any period.)

For linear autonomous system

\[
\begin{align*}
\dot{x} &= ax + by \\
\dot{y} &= cx + dy
\end{align*}
\]

periodic soln. iff roots of 
\[
\lambda^2 - (a+d)\lambda + (ad-bc) = 0
\]
are pure imaginary (in fact all solns. are periodic). Nonlinear auto. system very different.
\[ \begin{align*}
\dot{x} &= x + y - x (x^2 + y^2) \\
\dot{y} &= -x + y - y (x^2 + y^2)
\end{align*} \]

Only CP is \((0,0)\); system almost linear in neighborhood of origin, which is unstable special point. Might think all trajectories spiral out to infinity.

Introduce polar coordinates \(r, \theta\)

\[ x = r \cos \theta, \quad y = r \sin \theta \]

Now multiply first eq. by \( y \), second by \( y \) and add.

\[ \begin{align*}
\dot{x} \cdot y + \dot{y} \cdot y &= x^2 + x \cdot y - x^2 (x^2 + y^2) + \\\n&\quad - x^2 + y^2 - y^2 (x^2 + y^2) \\
\dot{y} \cdot y &= (x^2 + y^2) - (x^2 + y^2)^2
\end{align*} \]

\[ r^2 = x^2 + y^2 \]

\[ r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} \]

\[ \frac{dr}{dt} = r^2 (1 - r^2) \]
If $r > 1$, the direction of motion on the trajectory is inward.

If $r < 1$, the direction is outward.

Circle $r = 1$ has special significance.

Equation for $\Theta$. Multiply both sides by $x$, and substitute

\[ yx - xy = xy + \frac{z}{x} - xy(x^2 + y^2) \]

\[ x - xy + xy(x + y^2) \]

\[ = x^2 + y^2 \]

LHS is $r \cos \Theta \frac{d}{dt}(r \cos \Theta) - r \sin \Theta \frac{d}{dt}(r \sin \Theta)$

\[ - r \cos \Theta \frac{d}{dt}(\sin \Theta + \frac{r}{\cos \Theta}) \]

\[ = - r^2 \Theta \frac{d}{dt}(\sin \Theta + \cos \Theta) \]

\[ = - r^2 \frac{d^2}{dt^2} \Theta \]

Above reduces to $\frac{d^2}{dt^2} \Theta = -1$.

One solution of this system for $r$
and \( \Theta \) is

\[
\Theta = 1, \quad \Theta(t) = -t + t_0
\]

where \( t_0 \) is an arbitrary constant.

Other solutions found by separation of variables: if \( \gamma > \beta, \quad t \neq 1 \)

\[
\frac{1}{\gamma (1 - \gamma^2)} \frac{d\Theta}{dt} = 1
\]

Can use partial fractions to obtain

\[
\Theta(t) = \frac{1}{1 + c_0 e^{-2t}}, \quad \Theta(t) = -t + t_0
\]

c_0, t_0, constants.

Solution satisfying initial conditions

\( r = \rho, \quad \Theta = \alpha \) at \( t = 0 \) is given by

\[
\Theta(t) = \frac{1}{1 + [(\gamma^2 - 1)]e^{-2t}}, \quad \Theta(t) = -(t - \alpha)
\]

\( 0 < \gamma < 1, \quad \Theta(t) \to 1 \) from inside at \( t \to 0 \)

\( \gamma = 1, \quad \Theta(t) \to 1 \) from outside at \( t \to 0 \)
Trajectories spiral toward circle $r = 1$ at $t = \infty$.

A closed curve in phase plane which has unbounded curves spiraling toward it from the inside or outside as $t \to \infty$ is called a limit cycle. If all traj. spiral toward the LC,
it is stable. If traj. on one side spiral toward LC and on the other spiral away, it is unstable.
If traj. on both sides recede, not so.

The LC is unstable. (Linear periodic case neutrally stable)