Numerical Linear Algebra  April 12
Elementary Hermitian Matrices

\[ H(w) = E(w, w; k) = I - 2ww^H \text{ where } w^Hw = 1 \]

\[ \text{proj. } l = \text{span } \text{span } \text{span } \text{span } w_w \]

\[ R_x = x + 2 \left[ (I-P)x - x \right] = x - 2Px \]

\[ = x - 2 \langle x, w \rangle w \]

The operator \( R_x \) so defined "reflects the entire space through the ortho-complement of \( \text{span } w_w \)."
Finite dim. case
\[ R x = x - 2 (w^H x) w \]
\[ = x - 2 w w^H x \]
\[ = (I - 2 w w^H) x = H(w) x \]

R is represented by simple Hermitian.

If a and b are vectors of equal length, \( a^H a = b^H b \), and so that \( a^H b = b^H a \), i.e. is real, then it is possible to choose \( w \) so that \( H(w) a = b \).

Start with \( H = b \) and try to determine what \( w \) should be.

\[ (I - 2 w w^H) a = b \]
\[ a - 2 w w^H a = b \]
\[ a - b = 2 (w^H a) w \]
so \( w \) must be a multiple of \( a \cdot b \). Since \( \|w\| = 1 \), take
\[ w = \frac{a \cdot b}{\|a \cdot b\|}. \]
For \((*)\) to hold, one must have
\[ 1 = 2 \left( \frac{a \cdot b}{\|a \cdot b\|} \right) \cdot \frac{1}{\|a \cdot b\|} \]
\[ \|a \cdot b\|^2 = 2 \left[ a^\dagger a - b^\dagger b \right] \]
\[ a^\dagger a - b^\dagger b - b^\dagger b + b^\dagger b = 2a^\dagger a - 2b^\dagger a \]
which will hold if \( a^\dagger a = b^\dagger b \) and
\[ a^\dagger b = b^\dagger a \ (\text{real}) \].

Example: Let \( a = [0 \cdots 0 \alpha_k a_{k+1} \cdots a_n]^\dagger \)
and \( b = [0 \cdots 0 \varepsilon \cdots 0]^\dagger \) with \( \varepsilon \)
be determined. The condition
\[ a^\dagger b = b^\dagger a \] becomes \( a_k \varepsilon = \varepsilon a_k \)
\[ a_k \varepsilon = \varepsilon a_k \]
a condition (not the only one) which
which will insure this if \( \varepsilon = \beta a_k \)
with \( \beta \) real. To find \beta use the
condition \( b^* b = a^* a \):
\[
|\beta|^2 = \beta^2 |a_k|^2 = |a_k|^2 + \cdots + |a_n|^2
\]
\[
\beta = \pm \left\{ \frac{1}{|a_k|^2} \left( |a_k|^2 + \cdots + |a_n|^2 \right) \right\}^{1/2}
\]
assuming \( |a_k|^2 \neq 0 \). If \( a_k = 0 \), we can still be chosen so that
\[
H(w) a = \hat{b} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^T
\]
because \( a^* b = b^* a \) is immediate and
\[
\hat{\varepsilon} = \left\{ \frac{1}{|a_k|^2} + \cdots + |a_n|^2 \right\}^{1/2}
\]
Factorization by Elementary Transformations
\[
A = A^{(0)} \begin{bmatrix} a^{(0)}_1 & \bar{a}^{(0)}_2 & \cdots & \bar{a}^{(0)}_n \end{bmatrix} \neq 0
\]
a matrix of \( m \) rows and \( n \) columns
Let \( a^{(0)}_1 \) be the first nonzero column
and form \( H(w_1) a^{(0)}_1 \) so that \( H(w_1) a^{(0)}_1 \) is a multiple of \( \varepsilon \).
\[
A^{(1)} = H_1 A^{(0)}
\]
If any row below the first in \( A^{(1)} \) is not zero, let \( a^{(1)} \) be the first column such that \\
"the component of \( a^{(1)} \) orthogonal to \( e_1 \) is not zero. \\
Form \( H_2 = V H(w_2) \) so that the component of \\
\( a^{(1)} \) orthogonal to \( e_1 \) is reflected \\
into a multiple of \( e_2 \). If \( n = 2 \), \\
then \( H_2 = I \). \\
\[
A^{(2)} = H_2 A^{(1)}
\]
\[
\begin{pmatrix}
\cdots & \cdots \\
0 & 1
\end{pmatrix} = H_2 \cdot \\
\begin{pmatrix}
\cdots & \cdots \\
0 & \cdots
\end{pmatrix}
\]
It is important to note that the \\
multiple of \( e_1 \), which appears in \\
column \( j \) of \( A^{(1)} \), is left fixed by \\
\( H_2 \) since \( e_1 \) \( w_2 = 0 \).
Continuing this process, a matrix $A^{(p)}$ is formed such that $A = H A^{(p)}$, where $H = H_1 H_2 H_3 \ldots \ldots$ and either $p = m$ or else every row below the $p$th in $A^{(p)}$ is zero. It is not Hermitian, but it is unitary ($H^* = H^{-1}$).

If $p < m$, drop the zero rows in $A^{(p)}$ and the corresponding columns of $H$, denoting the results by $R$ and $W$ respectively. 

\[
A = WR
\]
\[
W^* W = I
\]
where $R$ has $p$ independent rows and $W$ has $p$ independent columns. If $A$ is non-singular, then $W$ is
vintage and \( \mathbf{R} \) is upper triangular. Moreover, this factorization is  

in a certain sense unique up to factors of unit modulus multiplying the columns of \( \mathbf{W} \).

\[
\mathbf{A} = \mathbf{H}_1^{-1} \mathbf{H}_2^{-1} \mathbf{H}_3 \mathbf{H}_1 \mathbf{A} \\
= \mathbf{H}_1^{-1} \mathbf{H}_2^{-1} \mathbf{H}_3^{-1} \mathbf{H}_3 \mathbf{H}_2 \mathbf{H}_1 \mathbf{A} \\
\quad \mathbf{W} \quad \rightarrow \quad \mathbf{R}
\]

\[
\mathbf{A} = \mathbf{W} \mathbf{R} \\
= \mathbf{Q} \mathbf{R}
\]

\[
\mathbf{A} = \mathbf{I} \mathbf{A} = \mathbf{L}(\mathbf{l}_1)^{-1} \mathbf{L}(\mathbf{l}_1) \mathbf{A} \\
= \mathbf{L}_1(\mathbf{l}_1)^{-1} \mathbf{L}(\mathbf{l}_1) \mathbf{A} \\
= \mathbf{L}_1^{-1} \mathbf{L}_2^{-1} \mathbf{L}_2 \mathbf{L}_1 \mathbf{A} \\
\quad \mathbf{L} \quad \rightarrow \quad \mathbf{U}
\]
\[ H^2 = (I - 2ww^H)(I - 2ww^H) \]
\[ = I - 2ww^H - 2ww^H + 4ww^Hww^H = I \]
\[ \overline{H = H^{-1} = H^*} \]
\[ H_1^1, A = \underbrace{H_1^1, H_2^2}_{Q}, \underbrace{H_2^2, H_1^1, A}_{R} \]
\[ (Q_1, Q_2^*) = Q_2^*, Q_1^* = Q_2^{-1}, Q_1^{-1} = (Q_1, Q_2)^{-1} \]