ODE II  
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Def. A point $y$ is an $\omega$-limit point of a solution (trajectory) $g(p,t)$ if there exists a sequence $t_1, t_2, \ldots \to +\infty$ such that $\lim_{n \to \infty} g(p, t_n) = y$. A point $y$ is called an $\alpha$-limit point if there exists a seq. $t_1, t_2, \ldots \to -\infty$ such that $\lim_{n \to \infty} g(p, t_n) = y$.

The set of all $\omega$-limit points of a given trajectory is called its $\omega$-limit set and denoted by $\Omega p$. The $\alpha$-limit set $\Lambda p$ of a given trajectory...
is the set of its $\alpha$-limit points. Both $\Omega_p$ and $\Lambda_p$ are closed sets. Thus, if $q$ is either an $w$- or $\alpha$-limit point of a traj. $g(p, t)$, then all other points of the trajectory $g(q, t)$ are also $w$- or $\alpha$-limit points respectively of the given traj. $g(p, t)$. In other words, both $w$- and $\alpha$-limit sets of a trajectory consist of whole trajectories.

Clarify trajectories according to properties of $\alpha$- and $w$-limit sets. A solution (or traj.) recedes in the positive direction if it has no $w$-limit points. A solution $g(p, t)$ is asymptotic in
The positive direction if there exist $w$-limit points, but they do not belong to this solution.

A solution $(p, t)$ is stable in the positive direction in the sense of Poisson if it has $w$-limit points which belong to this solution.

Symbol definitions: describe behavior of solutions as $t \to -\infty$.

1) Singular point as $P=p$, $q(p, t) = P$ is a trajectory. Every $CP$ is its own $\alpha$- as well as $w$-limit point, hence Poisson stable.

Set of singular points is closed: $x_1, x_2, \ldots \rightarrow y \rightarrow f(x_1), f(x_2), \ldots \rightarrow f(y)$

Moreover any closed set is the singular set for some dynamical system.
Ex. \[
\begin{align*}
\dot{x} &= -y + (x^2 + y^2 - 1) x \sin \left( \frac{1}{x^2 + y^2 - 1} \right) \\
\dot{y} &= x + (x^2 + y^2 - 1) y \sin \left( \frac{1}{x^2 + y^2 - 1} \right)
\end{align*}
\]
for \( x^2 + y^2 \neq 1 \) and \( x^2 + y^2 = 1 \)

In polar coordinates:
\[
\begin{align*}
\frac{dr}{dt} &= r \left( r^2 - 1 \right) \sin \left( \frac{1}{r^2 - 1} \right) \\
\frac{d\theta}{dt} &= 0 \quad r = 1
\end{align*}
\]
and \( \frac{d\theta}{dt} = \frac{1}{r} \)

In every neighborhood of the periodic solution
\[
x(t) = \cos (\Theta + t), \quad y(t) = \sin (\Theta + t)
\]
there are infinitely many periodic solutions
\[
x(t) = \Gamma_k \cos (\Theta + t), \quad y(t) = \Gamma_k \sin (\Theta + t)
\]
where \( \Gamma_k = \sqrt{1 + (2 \kappa t)} \) and satisfies
\[ \frac{1}{r^2 - 1} = 0. \] In each ring-shaped region between 2 consecutive circles, the trajectories are spirals approaching these two circles. Every one of these circles corresponds to a limit cycle.