Num. linear Algebra     April 17

Lecture 11 - Least Squares

\[ A x = b \] derived problem

\[ \min_x \| A x - b \|_2^2 \] generalized solution

e.g.: polynomial data-fitting

\[ x_1, \ldots, x_m \in \mathbb{C} \] \( m \) points

data \( y_1, y_2, \ldots, y_m \in \mathbb{C} \) at these points.

There is a unique interpolating poly. for the data of degree at most \( m-1 \):

\[ p(x) = c_{m-1} x^{m-1} + \cdots + c_1 x + c_0 \]

such that \( p(x_j) = y_j, \ j = 1, \ldots, m \)

Square Vandermonde system:

\[
\begin{bmatrix}
1 & x_1^2 & \cdots & x_1^{m-1} \\
1 & x_2^2 & \cdots & x_2^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_m^2 & \cdots & x_m^{m-1}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{m-1}
\end{bmatrix} =
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{bmatrix}
\]
Solve this nonsingular system for coefficients $c_j$. Why is it invertible?

(a) $\det \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} = (x_2 - x_1) \neq 0$ since $x_2 \neq x_1$.

(b) $\{1, I, I^2, I^3, \ldots, I^{m-1}\}$ where $I(x) = x$ on $[-1, 1]$ is linear independent.

Results are disappointing! Large oscillations near the ends of the interval artifact of interpolation process.

Poly LS Fitting. Reduce the degree of poly. Again $x_i, \ldots, x_m$ and $y_i, \ldots, y_m$, with vector $(n-1)$ deg. poly.

$p(x) = c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$
where $n < m$. Minimize the sum of squares of deviation from data:

$$
\sum_{j=1}^{m} 1_p(x_i) - y_i^j \cdot 2
$$

Sum of squares = square of norm of residual for rectangular Vandermonde system.

$$
\|e\|^2 = \|b - Ax\|^2
$$

$$
\begin{bmatrix}
1 & x_1 & x_1^{n-1} \\
1 & x_2 & x_2^{n-1} \\
\vdots & \vdots & \vdots \\
1 & x_m & x_m^{n-1}
\end{bmatrix}
\begin{bmatrix}
C_0 \\
C_1 \\
\vdots \\
C_{n-1}
\end{bmatrix}
= 
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{bmatrix}
$$

New poly. no longer interpolates but captures overall behavior of data better.

2.5 Theory $n$ unknowns, $m > n$

equation $A \in \mathbb{C}^{m \times n}$, $b \in \mathbb{C}^m$

solution $x \mapsto Ax = b \iff b \in \mathbb{R}(A)$
system is over determined, \( r = b - Ax \) \( \in \mathbb{C}^m \) is the residual and in general cannot be made equal to zero, so try minimizing \( \frac{1}{2} \| r \|_2^2 \).

Given \( A \in \mathbb{C}^{m \times n} \), \( b \in \mathbb{C}^m \) find \( x \in \mathbb{C}^n \) such that the residual \( \| b - Ax \|_2 \) is minimized.

**Orthogonal Projection & the Normal Equations.** (Look at geo. & Fund.)

\[
A x = b \\
A x = P b \\
A^{-1}(Pb) = \frac{1}{2} w | Aw = Pb \]
Pick the min. norm LS sol.  
\[ S \subset \mathbb{C}^m, \quad x_0 \in \mathbb{C}^m \]

\[ S + x_0 = \left\{ y + x_0 : y \in S \right\} \]

\[ x_0 \rightarrow x_0 \]