1) Show that if \( \{v_1, \ldots, v_n\} \) is a basis (not necessarily orthogonal) of a subspace \( V \), then the closest point \( v \) to \( y \) in \( V = \text{span}(\{v_1, \ldots, v_n\}) \) is given in terms of the Gramian matrix \( M_{ij} = \langle v_i, v_j \rangle \).

2) Let \( \mathcal{A} \) be the operator defined by

\[
\mathcal{A} u = -u''
\]

acting on the space \( C^2[0,1] \) of functions satisfying zero Dirichlet boundary conditions.

If \( k \) is a positive integer, then \( \lambda = k^2 \pi^2 \) is an eigenvalue with associated eigenfunction \( \sin(k\pi x) \), i.e.

\[
\mathcal{A} f = \lambda f \quad \Rightarrow \quad 0 = f'' + \lambda f
\]

Fix \( n \) and consider the corresponding discrete problem on a finite difference.
grid of $n-1$ points.

(Solving $-u''(x) = g(x)$)

\[ u''(x) = \frac{u_3 - 2u_2 + u_1}{(\Delta x)^2} = g(x) \]

\[ u''(x_3) \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -\alpha^2 \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \]

\[ \Delta x = \frac{1}{n} \]

\[ -n^2 \lambda \xi = \Lambda \xi \]

Solve this linear system.

Consider $-n^2 \lambda \xi = \Lambda \xi$, where $A$ is the matrix with ones on the sub- and superdiagonals and $-2$ on the main diagonal. The eigenvalues of the matrix $-n^2 A$ are given by

\[ \lambda_k = 2 \left( 1 - \cos \left( \frac{\pi k}{n} \right) \right), k = 1, \ldots, n-1 \]

with associated eigenvectors.
with associated eigenvectors
\[ (u_K) = \sin \left( \frac{\pi K}{n} \right), \quad K = 1, \ldots, n-1 \]

What happens in the limit as \( n \to \infty \)?

As \( n \) increases, any iterative method such as Jacobi (simultaneous relaxation) becomes less and less effective, especially when \( n \approx 100 \).

The method becomes useless.

(Recall the Jacobi that if \( A \) is strictly diagonally dominant, i.e.,
\[ \sum_{j \neq i} |A_{ij}| < |A_{ii}| \quad \text{for all } i, \]

\[ \|I - D^{-1}A\|_\infty = \max_i \left( \sum_{j \neq i} |A_{ij}|/|A_{ii}| \right) < 1, \]
\[ g(x) = D'(b - (A - D)x) \]
\[ = D'b + (I - D'A)x \]
\[ \text{is a contraction! Guaranteed convergence.} \]

Observation: The high frequency components of the input are quickly reduced by a few Jacobian sweeps, while the low frequency components of the error are reduced very slowly.

\[ f = \langle g', f \rangle + \cdots \]
Modify Jacobi by using several nested grids. This is the idea behind MULTIGRID methods. $O(n)$ FFT $O(n \log n)$