ODE 4
April 24

Columnwise Assignment 4 (i) a

\[
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
1 & 0 \\
-1 & 0
\end{bmatrix}
\]

Stability: equilibrium point must satisfy certain stability criterion to be very significant physically. Lyapunov: ad. unstable if nearby solutions stay nearby for all future time.

Let \( f \) be a map from an open set \( W \) of the vector space \( \mathbb{R}^n \) into \( \mathbb{R}^n \) and \( x \in W \) is a CP of \( f \)

\[
x' = f(x)
\]

\( x \) is stable if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) of \( f(x) \) such that any solution \( x(t) \)
with $x(0) \in U$, is defined and in $U$ for $t > 0$.

$U_1 \subset U$

**Def.** If $U_1$ can be chosen so that
\[
\lim_{t \to \infty} x(t) = \bar{x}
\]
then $\bar{x}$ is asymptotically stable.

**Def.** An equilibrium $x$ that is not stable is called unstable.

A **sink** (eigenvalue negative) is asymptotically stable, hence stable.

A **center** (pure imaginary eigenvalue) is stable but not asymptotically stable. Even small linear perturbation results in a sink or source: hyperbolicity is a generic property for linear flows.
Thus let \( W \subset \mathbb{R}^n \) be open and \( f: W \rightarrow \mathbb{R}^n \) be \( C^1 \). Suppose \( f(x) = 0 \) and \( x \) is a stable critical point of \( x' = f(x) \). Then no eigenvalue of \( DF(x) \) has positive real part.

Cor. A hyperbolic equilibrium (\( DF(x) \) has no eigenvalue with real part zero) is either unstable or asymptotically stable.

**Gradient Systems**: A gradient system on an open set \( U \subset \mathbb{R}^n \) is a dynamical system

\[
x' = -\nabla V(x)
\]

where \( V: U \rightarrow \mathbb{R} \) is \( C^2 \) and

\[
\nabla V = \left( \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right)
\]

is the gradient vector field \( \nabla V: U \rightarrow \mathbb{R}^n \). Note
$$DV(x)y = \langle \nabla V(x), y \rangle$$

(Because $DV(x)y = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(x)y_j)$

Let $\dot{V} : U \to \mathbb{R}^n$ be the derivative of $V$ along trajectories:

$$\dot{V}(x) = \frac{d}{dt} V(x(t)) \bigg|_{t=0}$$

Then $\dot{V}(x) \leq 0$ for all $x \in U$ and

$V(x) = 0$ iff $x$ is an equilibrium.

$$\frac{d}{dt} V(x(t)) \bigg|_{t=0} = DV(x)x' = \langle \nabla V(x), -\nabla V(x) \rangle = -|\nabla V(x)|^2$$

Consider $\bar{x}$ be an isolated minimum of $V$. Then $\bar{x}$ is an asymptotically stable equilibrium of the gradient system $\dot{x} = -\nabla V(x)$.

Pf: The function $x \mapsto V(x) - V(\bar{x})$
is a strict Lipschitz function for $\sqrt{x}$ in some neighborhood of $\sqrt{x}$.

Aside: Implicit Function Theorem

$x^2 + y^2 = 1$. Find $y'$. Solve explicitly for $y$.

(2) Implicit Left.

What can we solve for $y$ "as a function" of $x$?

$y = \pm \sqrt{1 - x^2}$

$q(x, y) = x^2 + y^2 - 1$.

Given $g = 5x$.

$y(x_0, y) = 0$.

$\frac{dy}{dx}(x_0, y) = \frac{x}{y}$.

Want to build $y$ such that:

$q(x, y(x)) = 0$.

$g_1(x, y(x)) + g_2(x, y(x)) y'(x) = 0$.

Continued . . .