AN INTRODUCTION
TO ANALYTIC FUNCTIONS
with theoretical implications

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UNIVERSITY OF HOUSTON
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AN INTRODUCTION TO ANALYTIC FUNCTIONS

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to

STUDENTS OF MATHEMATICS
The ten chapters of this guidebook are the basis of a two-semester course which my successive beginning classes in function-theory have helped me to develop at the University of North Carolina and the University of Houston. There are two features of this development which I regard as essential in mathematical training:

(1) The concise approach wherein attention is focussed on correct mathematical procedures as tools with which one builds upon a few simple facts about numbers, and

(2) The pedagogical device whereby the traditional student-teacher relationship is gradually replaced by that of investigators concurrently exploring a logical pattern of ideas.

Exercises, lemmas, and theorems (together with proofs thereof) are intended for classroom presentation and discussion. Insofar as student activity permits, the instructor expects to play the role of moderator -- rather than that of lecturer.

The reader is cautioned to be critical, and is expected to detect and correct for himself various misprints and misstatements of fact -- some initially inadvertent and some deliberate on my part. Occasionally, I have raised the question with a class as to whether or not some of these misstatements should be eliminated for subsequent classes: invariably, I have been answered in the negative.

In view of the foregoing comments on pedagogy, one should not expect to find in these notes any proof of any theorem. Likewise, I have chosen neither to include any bibliographical references, nor to label theorems with the names of mathematicians commonly associated therewith. From time to time, after theorems have been proved in class, the instructor may supply such information. It has been my frequent experience, however, that an impressive label on a theorem is likely to have the psychological effect of preventing a student from obtaining a simple contextual proof which is available to him.

I wish to thank my students, and my former colleagues, especially F. Burton Jones and W. M. Whyburn, and my teacher, H. S. Wall, for their encouragement to me during the development of this course.

J. S. M.

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INTRODUCTION

The words collection, family, and set are understood to be synonymous. Whereas we do not pretend to define the intuitive idea of a set, we describe some of the ideas associated with this notion:

(1) If S is a set then there exist objects P and Q such that P belongs to S and Q does not belong to S; if S is a set and P is an object then either P belongs to S or P is not a member of S.

(2) A subset of the set X is a set each member of which is a member of X; if each of X and Y is a set and X is a subset of Y and Y is a subset of X then X is Y; a proper subset of the set X is a subset of X which is not X.

(3) A degenerate set is a set with only one member; if P is an object then there is a set of which P is the only member; if P is an object and S is the set of which P is the only member then S is not P.

(4) If each of X and Y is a set then the sum of X and Y is the set to which P belongs only in case either P belongs to X or P belongs to Y; if each of X and Y is a set and there is a member of X which is a member of Y then the common part of X and Y is the set to which P belongs only in case P belongs to X and P belongs to Y.

We assume some familiarity with the intuitive idea of an ordered pair, and we understand it to be consistent with the following propositions:

(1) If each of A and B is an object then there is only one ordered pair of which A is the first term and B is the second term -- the notation \{A,B\} describes this ordered pair.

(2) If A and B are objects then \{A,B\} is not the ordered pair \{B,A\}.

(3) If the ordered pair \{A,B\} is the ordered pair \{C,D\} then A is C and B is D.

The symbol = denotes the word is and \# denotes the phrase is not, the symbols < and > respectively denote the phrases is less than and is greater than, and the symbols \(\leq\) and \(\geq\) respectively denote the phrases is not greater than and is not less than. If each of x and y is a number then the symbols \(x+y\) and \(xy\) respectively denote the value of the ordinary addition and multiplication operations at the ordered pair \{x,y\}.

Although a detailed development of the number-system is not considered properly a part of this course, we enumerate here a set of assumptions sufficient to characterize the system -- leaving to the reader such detail-work as his tastes and inclinations may dictate. Briefly, then, we consider the number-system as consisting of

a) a set of which the members are those objects called numbers and of which
the set of all counting numbers is a subset,

b) the addition and multiplication operations which are such that if \{x,y\} is an ordered number-pair then each of \(x+y\) and \(xy\) is a number,

c) the order relation to which \{x,y\} belongs only in case \(x < y\),
and
d) the following sixteen agreements (or axioms):

1. If each of \(x\), \(y\), and \(z\) is a number then \(x+(y+z) = (x+y)+z\).
2. The number 0 has the property that if \(x\) is a number then \(0+x = x\).
3. If \(x\) is a number then \(-x\) is a number and \(x+(-x) = 0\).
4. If \(x\) and \(y\) are numbers then \(x+y = y+x\).
5. If each of \(x\), \(y\), and \(z\) is a number then \(x(yz) = (xy)z\).
6. The number 1 has the property that if \(x\) is a number then \(1x = x\).
7. If \(x\) is a number different from 0 then \(\frac{1}{x}\) is a number and \(x\frac{1}{x} = 1\).
8. If \(x\) and \(y\) are numbers then \(xy = yx\).
9. The number 0 is not the number 1.
10. If each of \(x\), \(y\), and \(z\) is a number then \(x(y+z) = xy + xz\).
11. If \(x\) and \(y\) are numbers then either \(x < y\) or \(y < x\).
12. If each of \(x\), \(y\), and \(z\) is a number and \(x < y\) and \(y < z\) then \(x < z\).
13. If each of \(x\), \(y\), and \(z\) is a number and \(x < y\) then \(x+z < y+z\).
14. If each of \(x\), \(y\), and \(z\) is a number and \(x < y\) and \(0 < z\) then \(xz < yz\).
15. If \(S\) is a number-set and there is a number which is not less than any member of \(S\) then there is a least number which is not less that any member of \(S\).
16. The number 1 is a counting number and if \(x < 1\) then \(x\) is not a counting number; if \(y\) is a counting number then \(y+1\) is a counting number and, if \(y < z\) and \(z < y+1\), then \(z\) is not a counting number.

The foregoing framework is, of course, to be supplemented with the customary algebraic notation and terminology. Within the postulated system, one identifies the **positive numbers** as the set of all numbers greater than 0, the **negative numbers** as the set of all numbers less than 0, the **integers** as the set to which \(x\) belongs only in case either \(x\) is a counting number or \(x\) is 0 or \(-x\) is a counting number, and the **rational numbers** as the set to which \(x\) belongs only in case there is a positive integer \(p\) such that \(px\) is an integer.

One of the facts about the number-system is that if \(y\) is a negative number then there does not exist a number \(x\) such that \(x^2 = y\). This observation leads us to the ideas of Chapter 1, wherein we learn how to embed the number-system in a larger analytical structure. Each of the ten chapters is prefaced with a simple diagrammatic picture, accentuating a salient idea from that chapter and incensed to encourage the student to draw his own pictures and to develop his own geometric intuition.
After Chapter 10, there are four appendices and a supplement. The ideas presented there are intended to indicate extensions of the main development. In this sense, the appendices provide the framework of a second-year course -- Appendix 4 may very well appeal to earlier inclinations of the reader.

We shall endeavor to preserve the essentials of the English language, and, to this end, we make certain observations. A plural form of a noun or of a verb is not used unless it is applicable. In particular, since it is inconceivable that "two objects may be the same object," the statement that A and B are objects means that A denotes an object and B denotes an object and the object denoted by A is not the object denoted by B. On the other hand, if each of A and B denotes an object then the statement that A is B means that the object denoted by A is the object denoted by B. The statement that there is only one object having the property P means that there is an object having the property P and that there are not two such objects. The statement that the proposition A is true only in case the proposition B is true means that A is true in case B is true and in no other case, i.e., that A is equivalent to B. Finally, the statement that either A is true or B is true means that if A is not true then B is true.
**DEFINITION**

The statement that \( x \) is a **complex number** means that either \( x \) is a number or \( x \) is an ordered number-pair with second term different from zero. If each of \( \{a,b\} \) and \( \{c,d\} \) is an ordered number-pair and \( bd \neq 0 \) then

1. \( a + \{c,d\} = \{c,d\} + a = \{a+c,d\} \)
2. if \( b+d = 0 \), then \( \{a,b\} + \{c,d\} = a+c \)
3. if \( b+d \neq 0 \), then \( \{a,b\} + \{c,d\} = \{a+c,b+d\} \)
4. \( 0 \{c,d\} = \{c,d\} 0 = 0 \)
5. \( b \{c,d\} = \{c,d\} b = \{bc,bd\} \)
6. if \( ad+bc = 0 \), then \( \{a,b\} \{c,d\} = ac-bd \)
7. if \( ad+bc \neq 0 \), then \( \{a,b\} \{c,d\} = \{ac-bd, ad+bc\} \)

The symbol \( i \) is used to denote the complex number \( \{0,1\} \).

**REMARK**

If each of \( x \) and \( y \) is a complex number then each of \( x+y \) and \( xy \) is a complex number.

If \( \{a,b\} \) is an ordered number-pair then \( a+ib = a+bi \) and

1. if \( b = 0 \) then \( a+ib = a \)
2. but (2) if \( b \neq 0 \) then \( a+ib = \{a,b\} \).

**DEFINITION**

If \( \{a,b\} \) is an ordered number-pair then

1. \( \text{Re}(a+ib) = a \) and is called the **real part** of \( a+ib \)
2. \( \text{Im}(a+ib) = b \) and is called the **imaginary part** of \( a+ib \)
3. \( (a+ib)^\ast = a+i(-b) \) and is called the **conjugate** of \( a+ib \)
4. \( |a+ib| = (a^2+b^2)^{1/2} \) and is called the **modulus** of \( a+ib \)

**REMARK**

If \( x \) is a complex number then \( x + x^\ast = 2 \text{Re } x \), \( x = x^\ast + 2i \text{Im } x \), and the modulus of \( x \) (a nonnegative number) has the property: \( |x|^2 = x x^\ast = (\text{Re } x)^2 + (\text{Im } x)^2 \).

**ALGEBRAIC THEOREMS**

1. If each of \( x, y, \) and \( z \) is a complex number then \( x+(y+z) = (x+y)+z \).
2. If \( x \) is a complex number then \( 0+x = x \).
3. If \( x \) and \( y \) are complex numbers then \( x+y = y+x \).
4. If each of \( x \) and \( z \) is a complex number then there is only one complex number \( y \) such that \( x+y = z \) (we denote it by \( z-x \), or simply by \( -x \) in case \( z = 0 \)).
5. If each of \( x, y, \) and \( z \) is a complex number then \( x(yz) = (xy)z \).
(6) If $x$ is a complex number then $1x = x$.
(7) If $x$ and $y$ are complex numbers then $xy = yx$.
(8) If each of $x$ and $z$ is a complex number and $x \neq 0$ then there is only one complex number $y$ such that $xy = z$ (we denote it by $\frac{z}{x}$ or by $z/x$).
(9) If each of $x$, $y$, and $z$ is a complex number then $x(y+z) = xy + xz$.
(10) If each of $x$ and $y$ is a complex number then $(xy)^* = x^*y^*$.

REMARK

As an aid to the intuition we introduce certain geometric terminology: a point is a complex number, the real line is the set of all numbers, the number-plane is the set of all complex numbers, the right half-plane is the set to which $x$ belongs only in case $x$ is a point and $\text{Re} \, x > 0$, the left half-plane is similarly characterized by the condition that $\text{Re} \, x < 0$, and the upper half-plane and lower half-plane by the conditions $\text{Im} \, x > 0$ and $\text{Im} \, x < 0$, respectively. Moreover, by the distance from the point $x$ to the point $y$ we mean the number $|y-x|$; the following algebraic facts should be verified: if each of $x$ and $y$ is a point then
(1) $|x-y| = |y-x|$ and $|xy| = |x||y|$.
(2) $|x-y|^2 = |x|^2 - 2 \text{Re} (x^*y) + |y|^2$.
(3) $|x-y| \leq |x-z| + |z-y|$ for each point $z$.

DEFINITION

A relation is a set each member of which is an ordered pair, and if $f$ is a relation then the following five statements are true:
(1) the initial set of $f$ is the set to which $x$ belongs only in case $x$ is the first term of some member of $f$, and the final set of $f$ is the set to which $y$ belongs only in case $y$ is the second term of some member of $f$.
(2) if $S$ is a subset of the initial set of $f$ then the $f$-image of $S$, denoted by $f(S)$, is the set to which $y$ belongs only in case $y$ is the second term of some member of $f$ with first term belonging to $S$ -- if $f(S)$ is a subset of the set $T$ then $f$ is said to map $S$ into $T$, and is said to map $S$ onto $f(S)$.
(3) the statement that $g$ is an extension of $f$ means that $g$ is a relation of which $f$ is a subset -- if $T$ is the initial set of $g$ then $g$ is an extension of $f$ to $T$.
(4) if $S$ is a subset of the initial set of $f$ then a contraction of $f$ to $S$ is a subset of $f$ of which $S$ is the initial set.
(5) the inverse of $f$, denoted by $f^{-1}$, is the relation to which $\{x,y\}$ belongs only in case $\{y,x\}$ belongs to $f$.

If each of $g$ and $h$ is a relation and there is a member of the final set of $h$ which
belongs to the initial set of \( g \) then the \textbf{composite of} \( g \) \textbf{with} \( h \), denoted by \( g[h] \), is the relation to which \( \{x,z\} \) belongs only in case there is a member \( \{x,y\} \) of \( h \) such that \( \{y,z\} \) belongs to \( g \).

**EXERCISES**

1. If \( A \) is the relation to which \( \{u,v\} \) belongs only in case \( u \) and \( v \) are numbers and \( u < v \) then \( A \) is an \textbf{order relation} -- in the sense that
   a. if each of \( \{x,y\} \) and \( \{y,z\} \) is in \( A \) then \( \{x,z\} \) is in \( A \),
   b. if \( \{x,y\} \) is in \( A \) then \( \{y,x\} \) is not in \( A \), and
   c. if \( x \) and \( y \) are members either of the initial set of \( A \) or of the final set of \( A \) then either \( \{x,y\} \) is in \( A \) or \( \{y,x\} \) is in \( A \).

2. If \( B \) is the relation to which \( \{u,v\} \) belongs only in case \( u \) and \( v \) are points and \( \text{Re} \ u < \text{Re} \ v \) then \( B \) is a \textbf{partial-order relation} -- in the sense that
   a. if each of \( \{x,y\} \) and \( \{y,z\} \) is in \( B \) then \( \{x,z\} \) is in \( B \), and
   b. if \( \{x,y\} \) is in \( B \) then \( \{y,x\} \) is not in \( B \).

3. If \( C \) is the relation to which \( \{u,v\} \) belongs only in case each of \( u \) and \( v \) is a point and \( \text{Re} \ u = \text{Re} \ v \) then \( C \) is an \textbf{equivalence relation} -- in the sense that
   a. if each of \( \{x,y\} \) and \( \{y,z\} \) is in \( C \) then \( \{x,z\} \) is in \( C \), and
   b. if \( \{x,y\} \) is in \( C \) then \( \{y,x\} \) is in \( C \).

4. If \( C \) is an equivalence relation and \( D \) is the set to which \( \{x,S\} \) belongs only in case \( x \) is in the initial set of \( C \) and \( S \) is the set to which \( y \) belongs only in case \( \{x,y\} \) belongs to \( C \), then \( D \) is a \textbf{functional relation} -- in the sense that there are not two members of \( D \) having the same first term.

**DEFINITION**

A \textbf{function} (or \textbf{functional relation}) is a relation of which there are not two members having the same first term. If \( f \) is a function then

1. \( f \) is \textbf{reversible} provided there are not two members of \( f \) having the same second term,

2. \( f \) is said to be a function \textbf{from} its initial set \textbf{onto} its final set, and \textbf{from} its initial set \textbf{to} (or \textbf{into}) each set of which its final set is a subset, and

3. the \textbf{value of} \( f \) \textbf{at} \( x \), denoted by \( f_x \) or by \( f(x) \), is the second term of that member of \( f \) of which \( x \) is the first term.

**EXERCISES**

1. The set of all ordered point-pairs of the form \( \{x,ix\} \) is the set of all ordered point-pairs of the form \( \{-ix,x\} \), and is a reversible function which maps
the right half-plane onto the upper half-plane, the upper half-plane onto the left half-plane, and the left half-plane onto the lower half-plane.

(2) Let \( U \) be the set to which \( x \) belongs only in case \( x \) is a point and \( |x| < 1 \), \( P \) be the set of all points different from \(-1\), and \( f \) be the set of all ordered pairs \( \{x, y\} \) such that \( x \) is in \( P \) and \( y = \frac{1-x}{1+x} \); \( f \) is a function from \( P \) onto \( P \), and \( f(U) \) is the right half-plane.

(3) Let \( k \) be a point in \( U \) and \( t \) be the set of all ordered pairs \( \{x, y\} \) such that \( x \) is a point and \( k \neq 1 \) and \( y = \frac{x - k}{k \ast x - 1} \); \( t \) is a reversible function, \( t^{-1} = t \), and \( t(U) = U \); if \( B \) is the set of all points with modulus 1 then \( t(B) = B \).

(4) The set of all ordered point-pairs of the form \( \{x, x^2\} \) is not reversible.

(5) The contraction \( g \) of the function in exercise (4) to the right half-plane is reversible and the final set of \( g \) consists of all points \( z \) such that \( \text{Im} \ z \neq 0 \) or \( \text{Re} \ z > 0 \). (note: \( z^{1/2} \) denotes \( g^{-1}(z) \) if either \( \text{Im} \ z \neq 0 \) or \( \text{Re} \ z > 0 \).

**DEFINITION**

The statement that the function \( f \) is constant means that the final set of \( f \) has only one member: if \( k \) is the only member of the final set of \( f \), then we may denote \( f \) simply by the symbol \( k \) when the context is such as to preclude ambiguity; \( f \) is constant on the set \( S \) provided the contraction of \( f \) to \( S \) is constant. If each of \( f \) and \( g \) is a function of which the final set is a point-set, and there is a member of the initial set of \( f \) which belongs to the initial set of \( g \), then

(1) the functions \( f + g \), \( f - g \), and \( fg \) consist respectively of all ordered pairs of the form \( \{x, a+b\} \), \( \{x, a-b\} \), and \( \{x, ab\} \) such that \( \{x, a\} \) is in \( f \) and \( \{x, b\} \) is in \( g \).

(2) the function \( -g \) consists of all ordered pairs \( \{x, -b\} \) such that \( \{x, b\} \) is in \( g \).

(3) the function \( \frac{f}{g} \) consists of all ordered pairs of the form \( \{x, \frac{a}{b}\} \) such that \( \{x, a\} \) is in \( f \) and \( \{x, b\} \) is in \( g \) and \( b \neq 0 \).

A point-function is a function of which the initial set is a point-set and the final set is a point-set. The symbol \( I \) is used to denote the function which consists of all ordered point-pairs of the form \( \{x, x\} \), called the identity function.

**REMARK**

The symbols \( ii, \frac{1-i}{1+i}, \frac{i-k}{k \ast i - 1}, \) and \( i^2 \) denote respectively the functions discussed in the first four of the exercises immediately preceding this definition. Although we shall not use the following terminology, we let it be a matter of record that: a variable is a function, a real variable is a function of which the final set is a number-set, a complex variable is a function of which the final set is a point-set, and the statement that \( y \) is a function of \( x \) means that \( x \) is a function and \( y \) is a function and there is a function \( f \) such that \( y = f[x] \), i.e., \( y \) is the composite of \( f \) with \( x \).
DEFINITION

The statement that the point-set $S$ is bounded means that there is a number $r$ such that if $x$ is in $S$ then $|x| \leq r$. The point $z$ is a boundary-point of the point-set $S$ provided that if $b > 0$ then there is a point $x$ in $S$ such that $|x-z| < b$ and there is a point $y$ not in $S$ such that $|y-z| < b$ -- the boundary of $S$ is the set to which $z$ belongs only in case $z$ is a boundary-point of $S$.

REMARK

We recall the following property of the number-system: if $S$ is a number-set then

1) if there is a number which is not less than any number in $S$ then there is a least number which is not less than any number in $S$, and

2) if there is a number which is not greater than any number in $S$ then there is a greatest number which is not greater than any number in $S$.

EXERCISES

1) If $S$ is a bounded number set then there is a boundary-point of $S$.

2) If $B$ is the set of all points $x$ such that $|x| = 1$ then $B$ is the boundary of the unit disc $U$ (to which $z$ belongs only in case $z$ is a point and $|z| < 1$).

DEFINITION

A sequence is a function with initial set $S$ such that each member of $S$ is a nonnegative integer, 0 is in $S$, and if $m$ and $n$ are integers and $0 < m < n$ and $n$ is in $S$ then $m$ is in $S$; the notation $\{t_p\}^n_0$ indicates a sequence $t$ of which the initial set contains $n$ but does not contain any number greater than $n$. If $t$ is a point-sequence then

1) a cluster-point of $t$ is a point $x$ such that if $b > 0$ and $m$ is a positive integer there is a positive integer $n$ such that $|x - t(m+n)| < b$.

2) $t$ has the limit $y$ provided $y$ is a point and if $b > 0$ there is a positive integer $m$ such that if $n$ is a positive integer then $|y - t(m+n)| < b$.

3) $t$ converges provided that if $b > 0$ there is a positive integer $m$ such that if $n$ is a positive integer then $|t(m) - t(m+n)| < b$.

THEOREM 1

If $t$ is an infinite number-sequence with bounded final set then there is a number which is a cluster-point of $t$.

DEFINITION

If $f$ is a function from the number-set $A$ to a number-set then
(1) $f$ is increasing provided $A$ has two members and if $x$ and $y$ are members of $A$ and $x < y$ then $f(x) < f(y)$.

(2) $f$ is nondecreasing provided that if $x$ and $y$ are members of $A$ and $x < y$ then $f(x) \leq f(y)$.

(3) $f$ is decreasing provided $-f$ is increasing.

(4) $f$ is nonincreasing provided $-f$ is nondecreasing.

If $t$ is a sequence, then $s$ is a subsequence of $t$ provided there is an increasing sequence $u$, with final set a set of nonnegative integers, such that $s = t[u]$.

**EXERCISE**

If $t$ is a point-sequence and $x$ is a point, the following statements are equivalent:

(1) $x$ is a cluster-point of $t$.

(2) There is a subsequence of $t$ which has the limit $x$.

**THEOREM 2**

If $t$ is an infinite point-sequence with bounded final set then $t$ has a cluster-point.

**THEOREM 3**

If the point-sequence $t$ converges then $t$ has a limit.

**DEFINITION**

A limit-point of the point-set $S$ is a point $x$ such that if $b > 0$ then there is a point $y$ in $S$ such that $0 < |y - x| < b$. The point-set $S$ is closed provided there is no limit-point of $S$ which is not a member of $S$. The closure $\overline{S}$ of the point-set $S$ is the set to which $x$ belongs only in case either $x$ is in $S$ or $x$ is a limit-point of $S$.

**THEOREM 4**

If the infinite point-set $S$ is bounded then there is a limit-point of $S$.

Hint: if $S$ is an infinite set then there is an infinite sequence $t$ such that $t$ is reversible and the final set of $t$ is a subset of $S$.

**THEOREM 5**

If $S$ is a sequence and, for each nonnegative integer $n$, $S_n$ is a closed and bounded point-set of which $S_{n+1}$ is a subset, there is a point $x$ such that if $n$ is a nonnegative integer then $x$ belongs to $S_n$. 
DEFINITION

If $f$ is a point-function then

1. $f$ is **continuous at the ordered pair** $(p, q)$ provided $(p, q)$ belongs to $f$ and, for each positive number $c$, there is a positive number $b$ such that if $(x, y)$ belongs to $f$ and $|x-p| < b$ then $|y-q| < c$.

2. $f$ is **continuous** provided that if $(p, q)$ is in $f$ then $f$ is continuous at $(p, q)$.

3. $f$ is **continuous on $S$** provided $S$ is a subset of the initial set of $f$ and the contraction of $f$ to $S$ is continuous.

REMARK

The statement that the point-function $f$ is continuous at the point $p$ is sometimes used to mean that $p$ is in the initial set of $f$ and $f$ is continuous at $(p, f(p))$.

EXERCISES

1. If each of $f$ and $g$ is a point-function continuous on the point-set $S$ then each of $f+g$ and $fg$ is continuous on $S$.

2. If the point-function $f$ is continuous on the point-set $S$ and $0$ does not belong to $f(S)$ then $\frac{1}{f}$ is continuous on $S$.

3. If the point-function $g$ is continuous on $S$ and the point-function $f$ is continuous on $g(S)$ then $f[g]$ is continuous on $S$.

THEOREM 6

If the point-function $f$ is continuous on the closed and bounded point-set $S$ then

1. $f(S)$ is closed and bounded.

2. There is a point $x$ in $S$ such that if $y$ is in $S$ then $|f(y)| \leq |f(x)|$.

3. If $f$ is reversible then $f^{-1}$ is continuous on $f(S)$.

DEFINITION

If $f$ is a point-function with initial set $D$ then

1. $f$ is **uniformly continuous** provided that, for each positive number $c$, there is a positive number $b$ such that if each of $x$ and $y$ is in $D$ and $|x-y| < b$ then $|f(x)-f(y)| < c$.

2. $f$ is **uniformly continuous on $S$** provided $S$ is a subset of $D$ and the contraction of $f$ to $S$ is uniformly continuous.

THEOREM 7

If the point-function $f$ is continuous on the closed and bounded point-set $S$ then $f$ is uniformly continuous on $S$. 
DEFINITION

If $S$ is a subset of the initial set of the point-function $f$ and $f(S)$ is bounded then

1. $|f|_S$ denotes the least number $b$ such that if $x$ is in $S$ then $|f(x)| \leq b$, and is called the **modulus of $f$ on $S$**.

2. for each nonnegative number $b$, $C_f(S,b)$ denotes the least number $c$ such that if each of $x$ and $y$ is in $S$ and $|x-y| \leq b$ then $|f(x)-f(y)| \leq c$, and is called the **modulus of continuity of $f$ on $S$ relative to $b$**.

EXERCISES

1. If the point-function $f$ is uniformly continuous on $S$ and $c > 0$ then there is a positive number $b$ such that $C_f(S,b) < c$.

2. If the point-set $S$ has the property that each point-function which is continuous on $S$ is uniformly continuous on $S$, then $S$ is a closed and bounded point-set.

3. The **number-interval** $[0,1]$, to which $x$ belongs only in case $x$ is a number which is neither less than 0 nor greater than 1, is a closed and bounded point-set.

4. If $x$ is a point-function from $S$ into $T$ and $x(S)$ is bounded and $f$ is a point-function such that $f(T)$ is bounded then, if $b > 0$, $C_f[x](S,b) \leq C_f[T,C_x(S,b)]$.

5. If $\{s_p\}_{0}^{m}$ is a sequence and $m > 0$ and $\{t_p\}_{0}^{n}$ is a sequence of which $s$ is a subsequence and $s_0 = t_0$ and $s_m = t_n$, there is an increasing sequence $v$ with final set a set of nonnegative integers such that $v_0 = 0$ and $v_m = n$ and $s = t[v]$.

POSSIBLE PROJECT

Find an extension $\Omega$ of the complex number-system, embodying extensions of the addition and multiplication operations and of the definition of modulus, such that

a) each complex number belongs to $\Omega$,

b) if each of $x$ and $y$ is in $\Omega$ then each of $x+y$ and $xy$ is in $\Omega$,

c) if $x$ is in $\Omega$ then $|x|$ is a positive number unless $x = 0$,

d) the following postulates are satisfied:

1. If each of $x$, $y$, and $z$ is in $\Omega$ then $x+(y+z) = (x+y)+z$.

2. If $x$ is in $\Omega$ then $-x$ is in $\Omega$, $0+x = x$, and $x+(-x) = 0$.

3. If $x$ and $y$ are in $\Omega$ then $x+y = y+x$.

4. If each of $x$, $y$, and $z$ is in $\Omega$ then $1x = x$ and $x(yz) = (xy)z$.

5. If $x$ is a complex number and $y$ is in $\Omega$ then $xy = yx$.

6. If each of $x$, $y$, and $z$ is in $\Omega$ then $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$.

7. If each of $x$ and $y$ is in $\Omega$ then $|x+y| \leq |x|+|y|$ and $|xy| \leq |x||y|$.

8. Each convergent sequence with final set in $\Omega$ has a limit in $\Omega$.

{Such an $\Omega$ is sometimes called a complete normed algebra (over the complex numbers).}
If the system is bounded, then the decision is
which way to go.

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DEFINITION

If the number-interval \([a,b]\) lies in the initial set of the point-function \(x\), \(x\) is of bounded variation on \([a,b]\) in case there is a number \(v\) such that if \(\{t_p\}_{p=0}^n\) is a non-decreasing number-sequence, \(t_0 = a\), and \(t_n = b\), then \(\sum_{p=1}^n |x(t_p) - x(t_{p-1})| \leq v\), in which case \(\int_a^b |dx|\) denotes the least such \(v\) (the total variation of \(x\) on \([a,b]\)).

EXERCISE

If the point-function \(x\) is of bounded variation on \([a,b]\)

(1) and \(c > 0\), there is a nondecreasing number-sequence \(\{s_p\}_{p=0}^m\) such that \(s_0 = a\), \(s_m = b\), and if \(\{t_p\}_{p=0}^n\) is a nondecreasing number-sequence of which \(s\) is a subsequence and \(t_0 = a\) and \(t_n = b\) then \(\int_a^b |dx| - c < \sum_{p=1}^n |x(t_p) - x(t_{p-1})|\).

(2) and \(a < k < b\), then \(x\) is of bounded variation on \([a,k]\), \(x\) is of bounded variation on \([k,b]\), and \(\int_a^k |dx| + \int_k^b |dx| = \int_a^b |dx|\).

DEFINITION

A Stieltjes subdivision of the number-interval \([a,b]\) is a nondecreasing number-sequence \(\{s_p\}_{p=0}^{2m}\) such that \(s_0 = a\), \(m\) is a positive integer, and \(s_{2m} = b\); for such a sequence \(s\),

(1) the norm of \(s\), denoted by \(|s|\), is the least number \(r\) such that if \(p\) is an integer and \(1 \leq p \leq m\) then \(|s_{2p} - s_{2p-2}| \leq r\).

(2) a refinement of \(s\) is a Stieltjes subdivision \(\{t_p\}_{p=0}^{2n}\) of \([a,b]\) such that there is an increasing sequence \(u\) with final set a set of nonnegative integers such that \(u_0 = 0\), \(u_m = n\), and if \(p\) is an integer in \([0,m]\) then \(s(2p) = t(2u_p)\).

EXERCISES

(1) If \(t\) is a refinement of the Stieltjes subdivision \(s\) of \([a,b]\) then \(|t| \leq |s|\).

(2) If \(r\) and \(s\) are Stieltjes subdivisions of \([a,b]\) then there is a refinement of \(r\) which is a refinement of \(s\).

DEFINITION

If each of \(x\) and \(y\) is a point-function with initial set including \([a,b]\) then

(1) if \(\{s_p\}_{p=0}^{2m}\) is a Stieltjes subdivision of \([a,b]\) then \(\sum_{p=1}^m y(s_{2p-1})[x(s_{2p}) - x(s_{2p-2})]\) denotes the point

(2) the Stieltjes integral from \(a\) to \(b\) of \(y\) with respect to \(x\), \(\int_a^b y \, dx\), is a point \(z\) with the property that, for each positive number \(c\), there is a Stieltjes subdivision \(s\) of \([a,b]\) such that if \(t\) is a refinement of \(s\) then \(|z - \sum_t y \, dx| < c\).

REMARK

One might wish also to have \(\int_a^b dx \, y\) for \(x\) and \(y\) with final sets in an algebra \(\Omega\).
EXERCISES

(1) If each of $\int_a^b y \, dx$ and $\int_a^b X \, dx$ exists then
$$\int_a^b (y+X) \, dx = \int_a^b y \, dx + \int_a^b X \, dx.$$ (2) If each of $\int_a^b y \, dx$ and $\int_a^b X \, dx$ exists then
$$\int_a^b y \, d(x+X) = \int_a^b y \, dx + \int_a^b X \, dx.$$ (3) If $\int_a^b y \, dx$ exists and $w$ is a complex number then
$$\int_a^b y \, d(wx) = \int_a^b wy \, dx = w \int_a^b y \, dx.$$ (4) If $\int_a^b y \, dx$ exists, $y([a,b])$ is bounded, and $x$ is of bounded variation on $[a,b]$, then $|\int_a^b y \, dx| \leq |y|_{[a,b]} \int_a^b |dx|.$ (5) If $\int_a^b y \, dx$ exists and $a < k < b$ then
$$\int_a^k y \, dx + \int_k^b y \, dx = \int_a^b y \, dx.$$ (6) If each of $x$ and $y$ is a point-function and $\int_a^b y \, dx$ exists then
$$\int_a^b y \, dx + \int_a^b x \, dy = y(b)x(b) - y(a)x(a).$$

THEOREM 8

If $[a,b]$ is a number-interval on which the point-function $y$ is continuous and the point-function $x$ is of bounded variation then $\int_a^b y \, dx$ exists and, for each Stieltjes subdivision $s$ of $[a,b]$, $|\int_a^b y \, dx - \Sigma_s y \, dx| \leq C_y([a,b],|s|) \int_a^b |dx|.$

LEMMA 1

If $t$ is a refinement of the Stieltjes subdivision $s$ of $[a,b]$ then $|\Sigma_s y \, dx - \Sigma_t y \, dx| \leq C_y([a,b],|s|) \int_a^b |dx|.$

LEMMA 2

If $r$ and $s$ are Stieltjes subdivisions of $[a,b]$ then $|\Sigma_r y \, dx - \Sigma_s y \, dx| \leq (C_y([a,b],|r|) + C_y([a,b],|s|)) \int_a^b |dx|.$

DEFINITION

If $[a,b]$ lies in the initial set of the point-function $x$ and $\{s_p\}_{p=0}^{2m}$ is a Stieltjes subdivision of $[a,b]$ then $\Sigma_s |dx|$ denotes the number $\sum_{p=1}^{2m} |x(s_{2p}) - x(s_{2p-2})|.$

EXERCISES

(1) If $u$ is an increasing function from $[0,1]$ onto $[0,1]$ then $u$ is continuous.
(2) If \( v \) is a point-function, then \( v \) is a decreasing function from \([0,1]\) onto \([0,1]\) only in case \( 1-v \) is an increasing function from \([0,1]\) onto \([0,1]\).

(3) If the point-function \( x \) is of bounded variation on \([0,1]\) and \( u \) is an increasing function from \([0,1]\) onto \([0,1]\) then \( \int_0^1 |dx[u]| = \int_0^1 |dx| \).

\textbf{Hint:} if \( t \) is a Stieltjes subdivision of \([0,1]\) and \( r \) is a refinement of \( t \) and of \( u^{-1}[t] \) then \( u[r] \) is a refinement of \( t \) and of \( u[t] \), and \( \Sigma_t |dx[u]| = \Sigma_{u[r]} |dx| \).

(4) If the point-function \( x \) is of bounded variation on \([0,1]\) then \( x[1-I] \) is of bounded variation on \([0,1]\).

\textbf{Hint:} if \( \{t_p\}^{2^n}_{p=0} \) is a Stieltjes subdivision of \([0,1]\) then \( 1-t[2^n-I] \) is a Stieltjes subdivision \( s \) of \([0,1]\) such that \( \Sigma_t |dx[1-I]| = \Sigma_s |dx| \).

\textbf{THEOREM 9}

If \( x \) and \( y \) are point functions such that \( \int_0^1 y \, dx \) exists, \( u \) is an increasing function from \([0,1]\) onto \([0,1]\), and \( v \) is a decreasing function from \([0,1]\) onto \([0,1]\), then

\[ \int_0^1 y[u] \, dx[u] = \int_0^1 y \, dx = -\int_0^1 y[v] \, dx[v]. \]

\textbf{DEFINITION}

The statement that \( K \) is a \textbf{path} means that \( K \) is a set each member of which is a function from \([0,1]\) to the number-plane, and there is a member \( x \) of \( K \) such that

(1) \( x \) is continuous and of bounded variation on \([0,1]\), and

(2) \( y \) belongs to \( K \) only in case there is an increasing function \( u \) from \([0,1]\) onto \([0,1]\) such that \( y = x[u] \).

\textbf{REMARK}

If \( y \) is a member of the path \( K \) then

(1) \( y \) is continuous and of bounded variation on \([0,1]\), and

(2) \( z \) belongs to \( K \) only in case there is an increasing function \( v \) from \([0,1]\) onto \([0,1]\) such that \( z = y[v] \).

\textbf{DEFINITION}

If \( x \) is a member of the path \( K \) then

(1) \( K \) is a \textbf{path} from \( x(0) \) to \( x(1) \).

(2) \( K \) is \textbf{closed} provided \( x(0) = x(1) \).

(3) the \textbf{length} of \( K \), denoted by \( \lambda(K) \), is the number \( \int_0^1 |dx| \).

(4) \( K' \) denotes the point-set \( x([0,1]) \), called the \textbf{carrier} of \( K \).

(5) \( K \) is a \textbf{path} in \( S \) provided \( S \) is a point-set of which \( K' \) is a subset.

(6) \( -K \) denotes the path of which \( x[1-I] \) is a member.
EXERCISES

(1) If \( x \) and \( y \) are members of the path \( K \) and the point-function \( f \) is continuous on \( K' \) then \( \int_0^1 f[x] \, dx = \int_0^1 f[y] \, dy \).

(2) If \( x \) belongs to the path \( K \) then \( y \) belongs to \(-K\) only in case there is a decreasing function \( v \) from \([0,1]\) onto \([0,1]\) such that \( y = x[v] \).

(3) If \( x \) belongs to the path \( K \) and \( y \) belongs to \(-K\) and the point-function \( f \) is continuous on \( K' \) then \( \int_0^1 f[y] \, dy = -\int_0^1 f[x] \, dx \).

DEFINITION

If each of \( K_1 \) and \( K_2 \) is a path, the statement that \( K = K_1(+)K_2 \) means that there is a member \( x \) of \( K_1 \) and a member \( y \) of \( K_2 \) such that \( x(1) = y(0) \) and \( K \) is the path which contains the function \( z \) defined as follows: if \( 0 \leq t \leq \frac{1}{2} \) then \( z(t) = x(2t) \), but if \( \frac{1}{2} \leq t \leq 1 \) then \( z(t) = y(2t-1) \). If \( K \) is a path and \( f \) is a point-function with initial set including \( K' \) then

\[ \int_K f \]

denotes that point \( z \) with the property that if \( x \) is in \( K \) then \( z = \int_0^1 f[x] \, dx \), and is called the **Cauchy integral over the path \( K \) of the function \( f \)**.

REMARK

One might wish also to have \( \int f \) for a function \( f \) with final set lying in an algebra \( \Omega \).

EXERCISES

(1) \[ |\int f| \leq |f|_{K',\lambda(K)} \] provided \( f \) is continuous on the carrier of the path \( K \).

(2) If each of \( K_1 \) and \( K_2 \) is a path and \( K = K_1(+)K_2 \) then \( \lambda(K) = \lambda(K_1) + \lambda(K_2) \) and, if the point-function \( f \) is continuous on \( K' \),

\[ \int_K f = \int_{K_1} f + \int_{K_2} f \]

(3) If \( K \) is a path then \( K(+)K \) is a closed path.

DEFINITION

The point-function \( f \) has slope at the ordered pair \( \{p,q\} \) in case \( \{p,q\} \) belongs to \( f \) and there is only one point \( z \) with the property that, for each positive number \( c \), there is a positive number \( b \) such that if \( \{r,s\} \) is in \( f \) and \( 0 < |r-p| < b \) then \( \frac{s-q}{r-p} - z < c \) -- and in this case the point \( z \) is called the **slope of \( f \) at \( \{p,q\} \)**.

EXERCISE

If the point-function \( f \) has slope at \( \{p,q\} \) then \( p \) is a limit-point of the initial set
of $f$, and $f$ is continuous at $(p,q)$; if $p$ is a limit-point of the initial set of the point-function $f$ and $z$ is a point with the property that, if $c > 0$, there is a positive number $b$ such that if $(r,s)$ is in $f$ and $0 < |r-p| < b$ then $|\frac{s-q}{r-p} - z| < c$, then $f$ has slope $z$ at $(p,q)$.

**DEFINITION**

If $A$ is the initial set of the point-function $f$ and $B$ is the subset of $A$ to which $p$ belongs only in case $f$ has slope at $(p,f(p))$, then

1. the **derivative** of $f$, denoted by $f'$, is the point-function to which $(p,z)$ belongs only in case $p$ is in $B$ and $z$ is the slope of $f$ at $(p,f(p))$.

2. $\theta_f$ is a function from the set $A \times B$, to which $w$ belongs only in case $w$ is an ordered pair with first term in $A$ and second term in $B$, such that if $(x,y)$ is in $A \times B$ then $\theta_f(x,y) = 0$ and

$$f(x) - f(y) = f'(y)(x-y) + (x-y)\theta_f(x,y).$$

3. $f$ is said to **have a derivative at** $p$ provided $p$ belongs to the set $B$.

**EXERCISES (Differential Calculus)**

1. If the point-function has a derivative then $(-f)' = -f'$.

2. $\left(\frac{1}{x}\right)' = -\left(\frac{1}{x}\right)^2$ and $1'$ is the constant $1$.

3. If each of $f$ and $g$ is a point-function and $S$ is the set to which $p$ belongs only in case each of $f$ and $g$ has a derivative at $p$ and $p$ is a limit-point of the initial set of $f+g$, then the contraction of $f' + g'$ to $S$ is a subset of $(f+g)'$ and the contraction of $f'g + fg'$ to $S$ is a subset of $(fg)'$.

4. If each of $f$ and $g$ is a point function and $T$ is the set to which $p$ belongs only in case $g$ has a derivative at $p$, $f$ has a derivative at $g(p)$, and $p$ is a limit-point of the initial set of $f[g]$, the contraction of $f'[g]g'$ to $T$ lies in $f[g]'$.

**THEOREM 10 (Integral Calculus, Part 1)**

If $f$ is a point function and $K$ is a path from $A$ to $B$ in the initial set of $f$ and $f'$ is continuous on $K'$ then $\int_K f' = f(B) - f(A)$.

**LEMMA**

If $[a,b]$ is a subinterval of $[0,1]$ and $x$ is in $K$ and $c > 0$, there is an increasing sequence $s_n$ such that $s_0 = a$ and if $n$ is a nonnegative integer such that $s_n < b$ then $s_{n+1}$ is the least number which is not less than any number $u$ in $[s_n,b]$ such that $|\theta_f(x(u), x(s_n))| \leq c$,.
and the sequence $s$ has the following properties:

(1) If $n$ is a nonnegative integer and $s_{n+1} \leq b$ then $|\phi_f(x(s_{n+1}), x(s_n))| \leq c$.

(2) There is a positive integer $m$ such that $s_m = b$.

**DEFINITION**

If the initial set of the point-function $h$ includes the carrier of the path $K$, $h/K$ denotes the set to which $y$ belongs only in case there is a member $x$ of $K$ such that $y = h[x]$ -- $h/K$ is called the *image of* $K$ under $h$.

**EXERCISE**

If $K$ is a path and $h$ is a point function such that $h'$ is continuous on $K'$ then $h/K$ is a path, $(h/K)' = h(K')$, and $\lambda(h/K) \leq |h'|_{K'} \lambda(K)$.

**THEOREM 11 (Integral Calculus, Part 2)**

If each of $g$ and $h$ is a point-function and $K$ is a path such that $h'$ is continuous on $K'$ and $g$ is continuous on $h(K')$ then $\int_{h/K} g = \int_{K'} g[h] h'$.

**DEFINITION**

If each of $A$ and $B$ is a set, $\emptyset$ is a relation with initial set $A \times B$, each of $g$ and $h$ is a relation, and there is a member $\{p, x\}$ of $g$ and a member $\{p, y\}$ of $h$ such that $x$ is in $A$ and $y$ is in $B$, then the *composite of* $\emptyset$ *with* $\{g, h\}$, denoted by $\emptyset[g, h]$, is the relation to which $\{p, q\}$ belongs only in case there is a member $\{p, x\}$ of $g$ and a member $\{p, y\}$ of $h$ such that $\{(x, y), q\}$ belongs to $\emptyset$.

**REMARK**

If the point-function $f$ has a derivative at the point $z$ then $\phi_f[I, z]$ is a point-function and consists of $\frac{f - f(z)}{1 - z} - f'(z)$ together with the ordered pair $\{z, 0\}$ -- at which it is continuous.
DEFINITION

Two point-sets are **mutually separated** provided neither of them contains a point or a limit-point of the other. A point-set is **connected** provided it is not the sum of two mutually separated sets. A **component of the point-set** $S$ is a connected subset of $S$ which is not a proper subset of any connected subset of $S$.

EXERCISES

(1) The interval $[0,1]$ is connected.

(2) If the point-set $S$ contains two points and $S$ is connected, then each point of $S$ is a limit-point of $S$.

(3) If the point-set $S$ has the property that if $x$ and $y$ are points of $S$ then there is a connected subset of $S$ containing both $x$ and $y$, then $S$ is connected.

(4) If $x$ belongs to the point-set $S$ then $x$ belongs to a component of $S$.

THEOREM 12

If the point-function $f$ is continuous on the connected set $S$ then $f(S)$ is connected.

DEFINITION

If each of $A$ and $B$ is a point then the **linear path from** $A$ to $B$, denoted by $[A;B]$, is the path containing the contraction to $[0,1]$ of $(1-I)A + I B$. The point-set $S$ is **convex** provided that if $A$ and $B$ are in $S$ then $[A;B]$ is a subset of $S$. The statement that $K$ is a **polygonal path** means that there exists a point-sequence $\{A_p\}_{0}^{n}$ such that $K = \sum_{0}^{n} [A_p;A_{p+1}] (+)$, i.e., $K = S_n$ where $\{S_p\}_{0}^{n}$ is a sequence such that $S_0 = [A_0;A_1]$ and if $p$ is a nonnegative integer less than $n$ then $S_{p+1} = S_p (+) [A_{p+1};A_{p+2}]$.

EXERCISES

(1) $[0;1]$ is the set of all increasing functions from $[0,1]$ onto $[0,1]$.

(2) If $A$ and $B$ are points then $[B;A] = -[A;B]$ and $\lambda([A;B]) = |B-A|$.

(3) If $p$ is a point, the set of all points $x$ such that $|x-p| < 4$ is convex.

(4) If $K$ is a path and $A$ is a point then $K = K(+) [A;A]$.

(5) Every convex point-set is connected.

(6) If $K_1$ is a path from $A$ to $B$, $K_2$ is a path from $B$ to $C$, and $K_3$ is a path from $C$ to $D$, then $K_1(+)K_2(+)K_3 = (K_1(+)K_2)(+)K_3$.

DEFINITION

The point-set $S$ is **open** provided there is not a point in $S$ which is a limit-point of a set of points not in $S$. A **region** is a point-set which is open and connected.
EXERCISES

(1) If \( p \) belongs to the open point-set \( S \) then there is a positive number \( r \) such that if \( x \) is a point and \( |x-p| < r \) then \( x \) belongs to \( S \).

(2) Each component of an open point-set is a region.

THEOREM 13

If \( A \) and \( B \) are points in the region \( R \), there is a polygonal path from \( A \) to \( B \) in \( R \).

DEFINITION

An analytic function is a point-function of which the initial set is a region and which has slope at each of its members. The statement that the point-function \( f \) is

(1) analytic in \( R \) means that the contraction of \( f \) to \( R \) is an analytic function.

(2) analytic at \( z \) means that \( z \) belongs to a region in which \( f \) is analytic.

An entire function is an analytic function of which the initial set is the number-plane.

REMARK

One might wish further to consider analytic functions from regions to an algebra \( \Omega \).

THEOREM 14

If \( f \) is an analytic function and \( f' \) has only the value 0 then \( f \) is constant.

THEOREM 15

If \( R \) is a region and \( f \) is a continuous point-function with initial set \( R \) then the following two statements are equivalent:

(1) There is an analytic function \( g \) such that \( g' = f \).

(2) If \( K \) is a closed path in \( R \) then \( \int_{K} f = 0 \).

EXERCISE

If the point-function \( f \) is continuous on the interval \([a,b]\) then \( \int_{a}^{b} f \, dt = \int_{[a;b]} f \).

DEFINITION

If \( z \) is a point and \( r > 0 \) then the principal square contour with center \( z \) and radius \( r \), denoted by \( S_r(z) \), is the path

\[ [z+r-ir; z+r+ir] (+)[z-r+ir; z-r-ir] (+)[z-r-ir; z-r+ir] (+)[z+r+ir; z+r-ir] . \]

THEOREM 16

If \( z \) is a point and \( r > 0 \) and \( K = S_r(z) \) then \( \int_{K} \frac{1}{I - z} = 2\pi i \).
LEMMA 1

If \( h \) is the contraction to \([-1,1]\) of the function \( \frac{1+i}{1+ii} \) then

1. \( h \) is reversible with final set \( Q \) such that \( w \) belongs to \( Q \) only in case \( |w| = 1 \) and \( \text{Im } w \geq 0 \), and in this case \( h^{-1}(w) = \frac{w - i}{1 - iw} = \frac{\text{Re } w}{1 + \text{Im } w} \).

2. The number \( \pi \), which is defined to be \( \int_{-1}^{1} |dh| \), is \( 2 \int_{-1}^{1} \frac{1}{1+i} 2 \text{dI} \).

LEMMA 2

If \( K \) is one of the four paths indicated in defining \( S_{r}(z) \) then \( \int_{K} \frac{1}{1-z} = \frac{\pi}{2} i \).

COROLLARY

There is an analytic function which is not the derivative of an analytic function.

DEFINITION

If \( z \) is a complex number different from 0 then the principal argument of \( z \), denoted by \( \text{Arg } z \), is defined as follows:

1. If \( \text{Im } z = 0 \) and \( \text{Re } z > 0 \) then \( \text{Arg } z = 0 \).
2. If \( \text{Im } z = 0 \) and \( \text{Re } z < 0 \) then \( \text{Arg } z = \pi \).
3. If \( \text{Im } z > 0 \) and \( t = \frac{\text{Re } z}{|z| + \text{Im } z} \) then \( \text{Arg } z = 2 \int \frac{1}{1+i} 2 \text{dI} \).
4. If \( \text{Im } z < 0 \) then \( \text{Arg } z = -\text{Arg } z^{*} \).

EXERCISE

If \( -\pi < u \leq \pi \) then there is only one point \( w \) such that \( |w| = 1 \) and \( \text{Arg } w = u \).

DEFINITION

If \( S \) is a point-set and \( r > 0 \) then \( S+r \) denotes the point-set to which \( w \) belongs only in case there is a point \( z \) in \( S \) such that \( |w-z| \leq r \).

EXERCISE

If \( S \) is a closed and bounded set lying in the region \( R \) then there is a positive number \( r \) such that \( S+r \) is a subset of \( R \), and if \( |A-B| \leq 2r \) then \([A;B] \) lies in \( S+r \).

THEOREM 17

Suppose the point-function \( f \) is continuous on the region \( R \) and \( K \) is a path in \( R \). If \( y \) is in \( K \) and \( c > 0 \), there is a positive integer \( n \) such that if \( M = \Sigma_{1}^{n} [y(\frac{p-1}{n});y(\frac{p}{n})]^{(a)} \) then \( M \) is a polygonal path in \( R \) and \( | \int_{K} f - \int_{M} f | < c \).
DEFINITION

The statement that \( K \) is a \textbf{triangular path} means that there is a point-sequence \( \{A_p\}_{p=0}^2 \) such that \( K = [A_0; A_1; A_2] + [A_1; A_2; A_0] \); another notation is \( K = [A_0; A_1; A_2; A_0] \).

\[\text{THEOREM 18 (Triangle Theorem)}\]

Suppose \( A, B, \) and \( C \) are points, \( D \) is the point-set to which \( w \) belongs only in case there is a point \( P \) in \([B; C]\)' such that \( w \) is in \([A; P]'\), and the point-function \( f \) has a derivative at each point in \( D \). If \( K = [A; B; C; A] \) then \( \int_K f = 0 \).

\[\text{LEMMA 1}\]

The point-set \( D \) is closed, bounded, and convex.

\[\text{LEMMA 2}\]

Let \( r = \frac{A+B}{2} \), \( s = \frac{B+C}{2} \), and \( t = \frac{C+A}{2} \). There is a triangular path \( M \) in \( D \) such that

1. \( M \) is \([A; r; t; A] \) or \([r; B; s; r] \) or \([s; C; t; s] \) or \([t; r; s; t] \),
2. \( x \) is in \( K' \) and \( y \) is in \( M' \) then \( |x - y| \leq \lambda(M) \),
3. \( \lambda(M) = \lambda(K)/2 \) and \( \int_M f \leq 4 \int_K f \).

\[\text{LEMMA 3}\]

There is a sequence \( T \) such that \( T_1 = K \) and for each positive integer \( n \)

1. \( T_{n+1} \) is a triangular path in \( D \) and \( \lambda(T_{n+1}) = \lambda(K)/2^n \),
2. \( x \) is in \( T_n \) and \( y \) is in \( T_{n+1} \) then \( |x - y| \leq \lambda(T_{n+1}) \),
3. \( \int_{T_{n+1}} f \leq 4^n \int_K f \).

\[\text{LEMMA 4}\]

Let \( y \) be a point-sequence such that for each positive integer \( n \) \( y_n \) belongs to \( T_{n+1} \):

1. for each positive integer \( m \) and each positive integer \( n \) \( |y_m - y_{m+n}| \leq \lambda(T_{m+1}) \),
2. the limit \( z \) of \( y \) is in \( D \) and for each positive integer \( m \) \( |y_m - z| \leq \lambda(T_{m+1}) \),
3. if \( m \) is a positive integer and \( w \) is in \( T_{m+1} \) then \( |w - z| \leq \lambda(T_{m+1}) \),
4. if \( m \) is a positive integer then \( \int_{T_{m+1}} f \leq 4^m \int_{T_{m+1}} (I-z) \phi(I, z) \).

\[\text{THEOREM 19}\]

Suppose the point-function \( f \) is analytic in \( R \). If \( K \) is a closed polygonal path in \( R \) and there is a point \( V \) such that if \( P \) is in \( K' \) then \([V; P]' \) lies in \( R \), then \( \int_K f = 0 \).
THEOREM 20

If $z$ is a point in the region $R$, and the point-function $f$ is continuous on $R$ and analytic at each point in $R$ different from $z$, then the following statements are true:

1) If $A$, $B$, and $C$ are points such that if $P$ is in $[B;C]$ then $[A;P]$ lies in $R$ and $K = [A;B;C;A]$ then $\oint_{K} f = 0$.

2) If $K$ is a closed polygonal path in $R$ and there is a point $V$ such that if $P$ is in $K'$ then $[V;P]$ lies in $R$, then $\oint_{K} f = 0$.

THEOREM 21

Suppose $z$ is a point in the region $R$ and the point-function $f$ is continuous on $R$ and analytic at each point in $R$ different from $z$. If $K$ is a closed path in $R$ and there is a point $V$ such that if $P$ is in $K'$ then $[V;P]$ lies in $R$, then $\oint_{K} f = 0$.

LEMMA 1

If $P$ is in $K'$ there is a positive number $r$ such that if $|Q-P| < r$ then $[V;Q]$ lies in $R$.

LEMMA 2

There is a positive number $r$ such that if $P$ is in $K'$ and $|Q-P| < r$ then $[V;Q]$ lies in $R$.

THEOREM 22

Suppose $K$ is a path, the point-function $h$ is continuous on $K'$, and $f$ is a sequence such that, for each nonnegative integer $n$, $f_n$ is the point-function to which $\{z,w\}$ belongs only in case $z$ is a point not in $K'$ and

$$w = n! \oint_{K'} \frac{h}{(1-z)^{n+1}}.$$

If $n$ is a nonnegative integer then

$$f_n' = f_{n+1}.$$

THEOREM 23

Suppose the point-function $f$ is analytic in $R$, $K$ is a closed path in $R$, and there is a point $V$ such that if $P$ is in $K'$ then $[V;P]$ lies in $R$. There is a point in $R$ which is not in $K'$, and if $z$ is such a point then

$$\oint_{K} \frac{f}{1-z} = f(z) \oint_{K} \frac{1}{1-z}.$$

THEOREM 24

Suppose $K$ is a closed path and $g$ is the point-function to which $\{z,w\}$ belongs only in case $z$ is a point not in $K'$ and

$$w = \oint_{K} \frac{1}{1-z}.$$
(1) The function $g$ is constant on each component of the complement of $K'$.

(2) $g(z) = 0$ for each $z$ in the unbounded component of the complement of $K'$.

**COROLLARY**

If the point-function $f$ is analytic in the region $R$ then $f'$ is analytic in $R$.

**THEOREM 25**

If the entire function $f$ has bounded final set then $f$ is constant.

**THEOREM 26**

Suppose $n$ is a positive integer, $\{A_p\}_{p=0}^n$ is a point sequence such that $A_n \neq 0$, and $F$ is the point-function $A_0 + \sum_{p=1}^n A_p A_n^p$ (a polynomial of degree $n$).

1. $F$ is an entire function.
2. If $z$ is a point and $|z| > 1 + \sum_{k=0}^{n-1} |A_k/A_n|$ then $|F(z)| > |A_n| |z|^{n-1}$.
3. There is a point $z$ such that $F(z) = 0$.

**DEFINITION**

If $f$ is a point-function and $k$ a nonnegative integer, $f^{(0)} = f$ and $f^{(k+1)} = (f^{(k)})'$.

**THEOREM 27**

Suppose the point-function $f$ is analytic in $R$, $w$ is a point in $R$, $r$ is a positive number such that, if $z$ is a point and $|z-w| < 2r$, $z$ is in $R$, and $D$ is the component of the complement of $S_r(w)'$ to which $w$ belongs.

1. If either $|\text{Re}(z-w)| > r$ or $|\text{Im}(z-w)| > r$ then $z$ belongs to the unbounded component of the complement of $S_r(w)'$.
2. $D$ is the subset of $R$ to which $z$ belongs only in case $|\text{Re}(z-w)| < r$ and $|\text{Im}(z-w)| < r$.
3. If $P$ is in $S_r(w)'$ then $[w; P]'$ lies in $R$.
4. If $k$ is a nonnegative integer and $z$ is in $D$ then

\[
    f^{(k)}(z) = \frac{k!}{2\pi i} \int_{S_r(w)} \frac{f(z-w)}{(I-z)^{k+1}}.
\]

5. If $n$ is a positive integer and $z$ is in $D$ then

\[
    f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(z-w)}{k!} z \frac{(z-w)^k}{k!} + \frac{1}{2\pi i} \int_{S_r(w)} \frac{f(z-w)}{(I-w)^n} \frac{f(z-w)}{I-z}.
\]

6. If $0 < s < r$ and $c > 0$, there is a positive integer $m$ such that if $n$ is a positive integer and $z$ is a point such that $|z-w| < s$ then

\[
    |f(z) - \sum_{k=0}^{m+n} f^{(k)}(z-w)\frac{(z-w)^{k}}{k!}| < c.
\]
THEOREM 28

If $S$ is a subset of the region $R$ such that there is no limit-point of $S$ in $R$, and the point-function $f$ is continuous on $R$ and analytic at each point in $R$ which is not in $S$, then $f$ is analytic in $R$.

DEFINITION

A simple region is a region $R$ such that if $K$ is a closed path in $R$ and the point-function $f$ is analytic in $R$ then $\int_K f = 0$.

REMARK

The region $R$ is simple only in case it is true that each analytic function with initial set $R$ is the derivative of an analytic function.

EXERCISES

1. If $V$ is a point in the region $R$ such that for each point $P$ in $R \{V; P\}'$ lies in $R$ then $R$ is simple.

2. If $R$ is a simple region and $z$ is a point which does not belong to $R$ then, for each closed path $K$ in $R$, $\int_K \frac{1}{z-t} = 0$.

3. Every convex open point-set is a simple region.

4. There is a simple region $R$ such that the complement of $R$ has infinitely many components.

5. If two simple regions have a point in common then their sum is simple.

OBSERVATIONS

1. If the point-function $f$ is analytic in the region $R$ then the function $\phi_f$ is continuous on $R \times R$ -- in the sense that if $\{z,w\}$ is in $R \times R$ and $c > 0$ then there is a positive number $b$ such that if $\{x,y\}$ is in $R \times R$ and $|x-z| < b$ and $|y-w| < b$ then $|\phi_f(x,y) - \phi_f(z,w)| < c$.

2. Suppose $\Omega$ is such an algebra as indicated at the close of Chapter 2, and $\frac{1}{x}$ exists for each $x$ in $\Omega$ different from $0$, i.e., if $x$ is in $\Omega$ and $x \neq 0$ then there is a member $y$ of $\Omega$ such that $xy = yx = 1$. If $w$ is a member of $\Omega$ which is not a complex number then $\frac{1}{w-1}$ is an "analytic function from the plane to $\Omega$" with bounded final set.
If $f$ is a conformal mapping of the unit disk $D$ onto an annulus $A$, then:

- The function $f$ maps the boundary of the disk $S^1$ to the boundary of the annulus $S^2$.
- The function $f$ is a bijection between the disk $D$ and the annulus $A$. 
- The annulus $A$ is defined by $1 < |z| < r$ for some $r > 1$. 

The function $f$ can be represented by a Möbius transformation $f(z) = rac{az + b}{cz + d}$, where $a, b, c, d$ are complex numbers satisfying $ad - bc 
eq 0$ and $|a|^2 - |b|^2 = r^2$.
THEOREM 29

If $f$ is an analytic function with initial set $R$, the following are equivalent:

1. $R$ contains a limit-point of a set $S$ such that if $z$ is in $S$ then $f(z) = 0$.
2. There is a point $w$ such that, if $k$ is a nonnegative integer, $f^{(k)}(w) = 0$.
3. There is a point $w$ in $R$ and a positive number $b$ such that if $z$ is a point and $|z-w| < b$ then $f(z) = 0$.
4. If $z$ is a point in $R$ then $f(z) = 0$.

EXERCISE (Unique Extension Theorem)

If each of $f$ and $g$ is an analytic function which is an extension to the region $R$ of the function $h$ and $R$ contains a limit-point of the initial set of $h$, then $f$ is $g$.

COMMENT

In continuation of a comment at the close of Chapter 1, concerning unused terminology, it is believed that the following interpretation of common usage may be appropriate:

1. $w$ is a single valued analytic function of the complex variable $z$ provided there is an analytic function $f$ such that $w = f(z)$.
2. If the final set $S$ of the function $z$ lies in the initial set $R$ of the analytic function $f$ and $R$ contains a limit-point of $S$ and $w = f(z)$, then $\frac{dw}{dz}$ denotes $f'(z)$.

DEFINITION

The principal logarithmic function $L$ is the function to which $\{z, w\}$ belongs only in case $z$ is a point such that either $\text{Re } z > 0$ or $\text{Im } z \neq 0$, and $w = \int K \frac{1}{[1;z]}$.

EXERCISES

1. If $K$ is a path from $l$ to $z$ in the initial set of $L$ then $L(z) = \int K \frac{1}{I}$.
2. $L$ is analytic, $L'$ is a proper subset of $\frac{1}{I}$, but $L$ is not a proper subset of an analytic function.
3. If $z$ is in the initial set of $L$ and $r > 0$ then $L(r) + L(z) = L(rz)$.
4. Suppose $z$ is in the initial set of $L$, and $h$ is the function to which $\{u,v\}$ belongs only in case each of $u$ and $uz$ is in the initial set of $L$ and $v = L(u) + L(z) - L(uz)$.
   If $D$ is the component of the initial set of $h$ to which $l$ belongs then $z^*$ is in $D$ and $h(u) = 0$ for each $u$ in $D$.
5. If $z$ is in the initial set of $L$ then $L(z^*) = L(z^*)$, $\text{Re } L(z) = L(|z|)$, and $L(z) = L(|z|) + L(z/|z|)$.
6. If $|w| = 1$ and $\text{Im } w > 0$ then $L(w) = i \text{ Arg } w$. 
(7) The point \( z \) belongs to the final set of \( L \) only in case \(-\pi < \text{Im} \, z < \pi\).

(8) \( L[I^2 \frac{1-I}{1+I} [-I]] \) is a reversible analytic function from the unit disc \( U \) onto the final set of \( L \) which has slope 4 at \( \{0,0\} \).

**DEFINITION**

If \( f \) is a sequence each value of which is a point-function then

1. \( f \) **converges at** \( x \) provided \( x \) is a point and if \( b > 0 \) there is a positive integer \( m \) such that if \( n \) is a positive integer then \( |f_m(x) - f_{m+n}(x)| < b \); \( f \) has the **limit** \( y \) at \( x \) provided each of \( x \) and \( y \) is a point and if \( b > 0 \) there is a positive integer \( m \) such that if \( n \) is a positive integer then \( |y - f_{m+n}(x)| < b \).

2. \( f \) **converges on** \( S \) provided \( S \) is a point-set at each member of which \( f \) converges; \( f \) has the **limit** \( g \) on \( S \) provided that \( g \) is a point-function and \( S \) is a point-set and if \( x \) belongs to \( S \) then \( f \) has the limit \( g(x) \) at \( x \).

3. \( f \) **converges uniformly on** \( S \) provided \( S \) is a point-set and if \( b > 0 \) there is a positive integer \( m \) such that if \( x \) is in \( S \) and \( n \) a positive integer then \( |f_m(x) - f_{m+n}(x)| < b \).

4. \( f' \) is a sequence such that if \( n \) is a nonnegative integer then \( f'_n = f'_n \).

**EXERCISES**

1. Let \( f \) be a sequence such that, for each nonnegative integer \( n \), \( f_n = \frac{1}{n - 1} \).

2. If \( f \) is a sequence each value of which is a point-function and \( f \) converges on \( S \) then there is a point-function \( g \) such that \( f \) has the limit \( g \) on \( S \); if, moreover, \( f \) converges uniformly on \( S \), then \( f \) has the **limit** \( g \) **uniformly on** \( S \) -- in the sense that if \( b > 0 \) then there is a positive integer \( m \) such that if \( n \) is a positive integer then \( |g - f_{m+n}|_S < b \).

3. Suppose \( K \) is a path, \( f \) is a sequence each value of which is a point-function which is continuous on \( K' \), and \( f \) has the limit \( g \) uniformly on \( K' \); the point-function \( g \) is continuous on \( K' \) and \( \int g \) is the limit of the point-sequence \( \int f \) , i.e., of the sequence \( t_n \) such that if \( n \) is a nonnegative integer then \( t_n = \int f_n \).

**THEOREM 30**

Suppose \( R \) is a region, \( f \) is a sequence each value of which is a point-function which is analytic in \( R \), \( f \) has the limit \( g \) on \( R \), and \( f \) converges uniformly on each closed and bounded point-set lying in \( R \). The following statements are true:

1. The point-function \( g \) is analytic in \( R \).

2. \( f' \) has the limit \( g' \) on \( R \).

3. \( f' \) converges uniformly on each closed and bounded point-set lying in \( R \).
DEFINITION

The statement that \( f \) is a power-series means that there is a point \( w \) and an infinite point-sequence \( A \) such that \( f = \sum p (I-w)^p \), i.e., \( f \) is a sequence and \( f_0 \) is the constant \( A_0 \) with initial set the number-plane and if \( n \) is a positive integer then \( f_n \) is the point-function \( A_0 + \sum_{1}^{n} A_p (I-w)^p \); in this case, \( f \) is said to be a power-series about \( w \) with coefficient sequence \( A \).

REMARK

If \( w \) is in the initial set of the analytic function \( h \) then, by Theorem 27, there is a positive number \( s \) such that the power-series \( \sum \frac{h^{(p)}(w)}{p!} (I-w)^p \) has the limit \( h \) uniformly on the point-set to which \( z \) belongs only in case \( |z-w| < s \).

EXERCISES

1. If \( B \) is an infinite point-sequence with bounded final set then the power-series \( \sum \frac{B}{p!} I^p \) about 0 converges uniformly on each closed and bounded point-set.

2. If \( f \) is an analytic function and there is a number \( M \) such that if \( k \) is a nonnegative integer then \( |f^{(k)}(0)| < M \), then there is only one entire function of which \( f \) is a subset.

DEFINITION

The exponential function \( E \) is that entire function \( f \) such that \( f' = f \) and \( f(0) = 1 \).

EXERCISES

1. \( E \) is the limit on the number-plane of the power-series \( \sum \frac{1}{p!} I^p \) about 0.

2. If \( z \) is a point then \( E(z^*) = E(z)^* \) and \( E(z) = 1 + \int_{[0; z]} E \).

3. If each of \( w \) and \( z \) is a point then \( E(w)E(z) = E(w+z) \).

4. The contraction of \( E \) to the set of all numbers is an increasing function with final set the set of all positive numbers.

5. If \( z \) is a number then \( L(E(z)) = z \) and, if \( z > 0 \), \( E(L(z)) = z \).

6. If \( z \) is in the initial set of \( L \) then \( E(L(z)) = z \).

7. \( E(-i\pi) = -1 \) and \( E(i\pi) = 1 \).

8. If \( z \) is a point and \( m \) is an integer then \( E(z + 2m\pi i) = E(z) \).

9. If \( z \) is a point and \( E(z) = 1 \) then \( \frac{z}{2m\pi i} \) is an integer.

Hint: if not then there is an integer \( m \) such that \( -\pi < \frac{z}{2} + \pi - 2m\pi < \pi \).

10. If \( z \) is a point then \( |E(z)| = E(\text{Re } z) \).
DEFINITION

If w is a point and \( r > 0 \) then the **principal circular contour** with center w and radius r, denoted by \( C_r(w) \), is the path which contains the contraction to [0;1] of the function \( w + r E[(2I-1)\pi i] \).

EXERCISES

(1) \( C_1(0) = E[-i\pi; i\pi] \) and \( \lambda(C_1(0)) = 2\pi \).

(2) If w is a point and \( r > 0 \) then \( C_r(w) = (w + ri)/C_1(0) \).

(3) If each of z and w is a point, \( r > 0 \), and \( |z-w| = r \), z is in \( C_r(w) \).

(4) If each of z and w is a point, \( r > 0 \), and \( |z-w| \neq r \), z is not in \( C_r(w) \)
and \( \frac{1}{2\pi i} \int_{C_r(w)} \frac{1}{1-z} \) is 0 or 1, accordingly as \( |z-w| > r \) or \( |z-w| < r \).

THEOREM 31

If K is a path from A to B and h is a point-function such that \( h' \) is continuous on \( K' \) and 0 is not in \( h(K') \), then \( E(\int_K \frac{h'}{h}) = \frac{h(B)}{h(A)} \).

DEFINITION

If K is a closed path and z is a point not in \( K' \) then the **winding number** of K about z, denoted by \( W(K,z) \), is the integer \( \frac{1}{2\pi i} \int_K \frac{1}{1-z} \). A **contour** is a closed path K such that if z is a point not in \( K' \) then \( W(K,z) \) is either 0 or 1.

REMARK

If the point-function f is analytic in R, K is a closed path in R, and P is a point not in \( K' \), then \( W(f/K,P) = \frac{1}{2\pi i} \int_K \frac{f'}{f - P} \).

THEOREM 32

If the point-function f is analytic in R, w is a point in R, and r is a positive number such that the disc \( D_r(w) \), to which z belongs only in case z is a point and \( |z-w| < r \), lies in R, then the power-series \( \sum_{p=0}^{\infty} \frac{f(p)(w)}{p!}(1-w)^p \) about w has the limit f on \( D_r(w) \) and converges uniformly on each closed point-set which lies in \( D_r(w) \).

DEFINITION

If w is a point then \( I^w \) denotes an analytic function which includes \( E[wL] \) but is not a proper subset of an analytic function, \( (w)_0 = 1 \), and if p is a nonnegative integer then \( (\frac{w}{p+1}) = \frac{w-p}{p+1} (\frac{w}{p}) \).
EXERCISE

If \( w \) is a point, the power-series \( \sum \frac{w^p}{p!} \) has the limit \( (1+i)^w = 1^{w[1+i]} \) on \( U \), and converges uniformly on each closed point-set lying in \( U \).

THEOREM 33

Suppose \( g \) is an analytic function with initial set \( R_1, R_2 \) is the set to which \( z \) belongs only in case \( z^* \) is in \( R_1, R_3 \) is the common part of \( R_1 \) and \( R_2 \), and \( R \) is the sum of \( R_1 \) and \( R_2 \). The following statements are true:

1. There is an analytic function \( h \) such that if \( z \) is in \( R_2 \) then \( h(z) = g(z^*)^* \).
2. Each component of \( R_3 \) is a region which contains a number.
3. If there is a component \( D \) of \( R_3 \) and a subset \( S \) of \( D \) with a limit-point in \( D \) such that if \( z \) is in \( S \) then \( g(z^*)^* = g(z^*) \), then there is an analytic function \( f \) which is an extension of \( g \) to \( R \).

DEFINITION

The sine function \( S \) is the entire function \( \frac{E[iI] - E[-iI]}{2i} \), the cosine function \( C \) is the entire function \( \frac{E[iI] + E[-iI]}{2} \), and the tangent function \( T \) is the analytic function \( \frac{S}{C} \).

EXERCISES

1. If \( z^* = z \) then \( C(z) = \text{Re} E(iz) \) and \( S(z) = \text{Im} E(iz) \).
2. If \( |w| = 1 \) and \( u = \text{Arg} w \) then \( C(u) = \text{Re} w \) and \( S(u) = \text{Im} w \).
3. \( C[\frac{\pi}{2} - I] = S \).
4. \( S(z) = 0 \) only in case \( \frac{z}{\pi} \) is an integer.
5. \( T[I + \pi] = T \).
6. \( S[2I] = 2SC \) and \( C[2I] = C^2 - S^2 \).
7. \( S' = C^2 \) and \( C' = -S \) and \( T' = 1 + T^2 \).
8. If \( A \) is the function to which \( \{z, w\} \) belongs only in case \( z \) is a point such that either \( \text{Re} z \neq 0 \) or \( -1 < \text{Im} z < 1 \), and \( w = \int_0^z \frac{1}{1 + I^2} \), \( A^{-1} \) is a subset of \( T \).
DEFINITION

The point-sequence \( t \) converges absolutely provided that if \( b > 0 \) there is a positive integer \( m \) such that if \( n \) is a positive integer then \( \sum_{m+1}^{m+n} |t_p - t_{p-1}| < b \). If \( f \) is a sequence each value of which is a point-function then

1. \( f \) converges absolutely at \( x \) provided \( x \) is a point and if \( b > 0 \) there is a positive integer \( m \) such that if \( n \) is a positive integer \( \sum_{m+1}^{m+n} |f(x) - f_{p-1}(x)| < b \).

2. \( f \) converges absolutely on \( S \) provided \( S \) is a point-set at each member of which \( f \) converges absolutely.

THEOREM 34

Suppose \( A \) is an infinite point-sequence and \( S \) is the power-series \( \sum A_p (I-w)^p \) about \( w \).
If \( z \) is a point different from \( w \), \( S \) converges at \( z \), and \( D \) is the disc to which \( x \) belongs only in case \( |x-w| < |z-w| \), then \( S \) converges absolutely on \( D \) and converges uniformly on each closed point-set lying in \( D \).

THEOREM 35

Suppose \( A \) is an infinite point-sequence and \( S \) is the power-series \( \sum A_p (I-w)^p \) about \( w \).
One of the following statements is true:

1. \( S \) is totally divergent, i.e., \( S \) converges only at \( w \).
2. \( S \) is totally convergent, i.e., \( S \) converges at every point.
3. \( S \) has a radius of convergence, i.e., there is a positive number \( r \) such that \( S \) converges at each point \( z \) such that \( |z-w| < r \) and \( S \) does not converge at any point \( z \) such that \( |z-w| > r \). (In this case, \( r \) is called the radius of convergence of \( S \)).

THEOREM 36

Suppose \( A \) is an infinite point-sequence and \( S \) is the power-series \( \sum A_p (I-w)^p \) about \( w \), and \( s \) is a number-sequence such that if \( n \) is a positive integer then \( s_n = |A_n|^{1/n} \).
The three cases described in Theorem 35 are characterized respectively as follows:

1. \( S \) is totally divergent in case the final set of \( s \) is not bounded.
2. \( S \) is totally convergent in case \( s \) has the limit 0.
3. \( S \) has the radius of convergence \( r \) in case the final set of \( s \) is bounded, \( s \) does not have the limit 0, and \( \frac{1}{r} \) is the greatest number which is a cluster-point of \( s \).

EXERCISE

If \( 0 < m < M \), \( w \) is a point, \( A \) is the set of all points \( z \) such that \( m < |z-w| < M \), \( S \) is the set of all points \( z \) such that \( L(m) < \Re z < L(M) \), and \( h = w + E \), then \( h \) maps \( S \) onto \( A \) and, if \( m < r < R < M \),
THEOREM 37 (Annulus Theorem)

Suppose $0 < m < M$, $w$ is a point, and the point-function $f$ is analytic in the annulus $A$, to which $z$ belongs only in case $z$ is a point and $m < |z-w| < M$.

1. If $m < r < R < M$ then

$$\int_{C_R(w)} f = \int_{C_r(w)} f.$$ 

2. If $D_1$ is the disc to which $z$ belongs only in case $|z-w| < \frac{1}{m}$ and $D_2$ is the disc to which $z$ belongs only in case $|z-w| < M$ then there is only one ordered pair $(g, h)$ such that $g$ is an analytic function with initial set $D_1$, $h$ is an analytic function with initial set $D_2$, $g(w) = 0$, and if $z$ belongs to $A$ then $f(z) = g(z) + \frac{1}{z-w} + h(z)$.

3. There is a function $n$ from the set of all integers such that if $m < r < M$ and $n$ is an integer then

$$b_n = \frac{1}{2\pi i} \int_{C_r(w)} \frac{f}{(I-w)^{n+1}};$$

if, moreover, $(g, h)$ is the ordered pair determined in part (2), then $b_0 = h(w)$ and for each positive integer $n$

$$b_{-n} = \frac{g(n)(w)}{n!} \quad \text{and} \quad b_n = \frac{h(n)(w)}{n!}.$$ 

REMARK

The representation of $f$ in the annulus $A$, in terms of $g$ and $h$ as in Theorem 37 (2,3), is often written as $f = \sum_{-\infty}^{+\infty} b_p (I-w)^p$ -- to which, in accordance with Theorem 32, one may attach the following interpretation: if $S$ is a closed set lying in the annulus $A$ and $c > 0$ then there is a positive integer $n$ such that if each of $u$ and $v$ is a positive integer and $z$ is in $S$, $|f(z) - \sum_{-n-u}^{n+v} b_p (z-w)^p| < c$. In the notation of Theorem 37, in case there is an entire function of which $g$ is a subset then, of course, there is an analytic extension of $f$ to a set including those points $z$ such that $0 < |z-w| < M$, and in this case $b_{-1}$ is called the residue of $f$ at $w$, and is seen to be that complex number $k$ such that the contraction of $f - \frac{k}{I-w}$ to $A$ is the derivative of a function.

EXERCISE

In the sense of the preceding Remark, if the analytic functions $f$ and $F$ have the representations $\sum_{-\infty}^{+\infty} b_p (I-w)^p$ and $\sum_{-\infty}^{+\infty} B_p (I-w)^p$, respectively, in $A$ and $m < r < M$,

$$\frac{1}{2\pi i} \int_{C_r(w)} \frac{fF}{I-w} = \frac{1}{2\pi} \int_{[-\pi; \pi]} f[w+rE[i\Omega]]F[w+rE[i\Omega]]^* = \sum_{-\infty}^{+\infty} b_p B_p r^{2p}.$$
THEOREM 38

If R is a region and \( \{w_p\}_{p=0}^n \) is a reversible point sequence with final set S lying in R and the point-function \( f \) is analytic in R-S -- to which \( z \) belongs only in case \( z \) is a point in R which is not in S -- then there is only one sequence \( \{g_p\}_{p=0}^{n+1} \) such that

1. if \( p \) is an integer in \([0,n]\) then \( g_p \) is an entire function and \( g_p(w_p) = 0 \),
2. \( g_{n+1} \) is an analytic function with initial set R, and
3. if \( z \) belongs to R-S then \( f(z) = \sum_{p=0}^{n} g_p(w_p + \frac{1}{z-w_p}) + g_{n+1}(z) \).

THEOREM 39 (Residue Theorem)

Suppose R is a simple region and \( \{w_p\}_{p=0}^n \) is a reversible point sequence with final set S lying in R and the point-function \( f \) is analytic in R-S. If, for each nonnegative integer \( p \) not greater than \( n \), \( k_p \) is the residue of \( f \) at \( w_p \) and \( K \) is a closed path in R-S then

\[
\frac{1}{2\pi i} \oint_{K} f = \sum_{p=0}^{n} k_p W(K,w_p).
\]

DEFINITION

The point-function \( f \) has order \( j \) at the point \( w \) provided that \( w \) is a limit-point of the initial set of \( f \), that there is a number \( v \) with the property

\((*)\) there exist positive numbers \( r \) and \( M \) such that if \( z \) is in the initial set of \( f \) and \( 0 < |z-w| < r \) then \( |f(z)| \leq |z-w|^v M \) (*),

and that \( j \) is the least number which is not less than any such number \( v \).

EXERCISE

The point-function \( I^{1/2} \) has order \( \frac{1}{2} \) at the point \( 0 \).

THEOREM 40

If the point-function \( f \) is analytic in R and \( w \) is a point in R,

1. \( f \) does not have negative order at \( w \).
2. if \( f \) does not have order at \( w \) then \( f(z) = 0 \) for each \( z \) in R.

3. if \( f \) has order \( j \) at \( w \) then \( j \) is an integer and there is a function \( g \), analytic in R, such that \( g(w) \neq 0 \) and \( f(z) = (z-w)^j g(z) \) for each \( z \) in R different from \( w \).

THEOREM 41

Suppose \( g \) is an entire function, \( w \) is a point, and each of \( v, R, \) and \( M \) is a positive number such that if \( z \) is a point and \( |z-w| > R \) then \( |g(z)| \leq |z-w|^v M \). If \( n \) is an integer greater than \( v \) then \( g^{(n)}(w) = 0 \) so that either \( g \) is constant or \( g \) is a polynomial of degree not greater than \( v \).
THEOREM 42

If \( w \) is a point in the region \( R \) and the point-function \( f \) is analytic at each point in \( R \) different from \( w \) and \( f \) has order \( j \) at \( w \), then \( j \) is an integer and there is a function \( g \), analytic in \( R \), such that \( g(w) \neq 0 \) and \( f(z) = (z-w)^j g(z) \) if \( z \) is in \( R \) and \( z \neq w \).

THEOREM 43

Suppose \( R \) is a region and \( \{ w_p \}_{p=0}^n \) is a reversible sequence with final set \( S \) lying in \( R \) and the point-function \( f \) is analytic in \( R-S \). If, for each nonnegative integer \( p \) not greater than \( n \), \( j_p \) is the order of \( f \) at \( w_p \) then

1. There is a function \( g \), analytic in \( R \), such that \( g(S) \) does not contain \( 0 \) and, for each point \( z \) in \( R-S \), \( f(z) = g(z) \prod_{p=0}^n (z-w_p)^{j_p} \), and

2. If \( K \) is a closed path in \( R-S \), \( 0 \) is not in \( f(K') \), and \( R \) is simple, then

\[
\frac{1}{2\pi i} \oint_K \frac{f'}{f} = \sum_{p=0}^n j_p W(K, w_p).
\]

DEFINITION

If \( w \) is a point in the region \( R \) and the point-function \( f \) is analytic at each point in \( R \) different from \( w \), then

1. \( w \) is a zero of \( f \) provided \( f \) has positive order at \( w \).

2. \( w \) is a pole of \( f \) provided \( f \) has negative order at \( w \).

3. \( w \) is an essential singularity of \( f \) if there do not exist a number \( \nu \) and positive numbers \( r \) and \( M \) such that if \( z \) is in \( R \) and \( 0 < |z-w| < r \) then \( |f(z)| \leq |z-w|^\nu M \).

THEOREM 44

Suppose \( w \) is a point in the region \( R \), the point-function \( f \) is analytic at each point in \( R \) different from \( w \), and \( w \) is an essential singularity of \( f \). If \( P \) is a point and \( b \) and \( r \) are positive numbers then there is a point \( z \) in \( R \) such that \( 0 < |z-w| < r \) and \( |f(z) - P| < b \).

COROLLARY

If the entire function \( g \) is not constant and \( g \) is not a polynomial then, for each point \( P \) and each positive number \( r \) and each positive number \( b \), there exists a point \( z \) such that \( |z| > r \) and \( |g(z) - P| < b \).

EXERCISE

\( 0 \) is an essential singularity of \( E[-1/\mathbb{I}^2] \), but \( \{0,0\} \) together with the contraction of \( E[-1/\mathbb{I}^2] \) to the set of all numbers different from \( 0 \) is "infinitely differentiable."
THEOREM 45 (Open Mapping Theorem)

If the point-function \( f \) is analytic and \( R \) is a region lying in the initial set of \( f \), then either \( f \) is constant or \( f(R) \) is a region.

**Hint:** if \( f \) is not constant and \( w \) is in \( R \) then there exists a positive integer \( n \) and a point-function \( g \), analytic in \( R \) such that \( g(w) \neq 0 \) and \( f(z) - f(w) = (z-w)^n g(z) \) for each \( z \) in \( R \) -- let \( r \) be a positive number such that if \( |z-w| \leq r \) then \( z \) is in \( R \) and \( g(z) \neq 0 \), and show that \( f(D_r(w)) \) includes a component of the complement of \( f(C_r(w)) \).

**EXERCISES**

1. If \( f \) is a nonconstant analytic function with initial set \( R \) and \( S \) is a bounded open set such that \( S \) lies in \( R \) then \( |f(z)| < |f|_S \) for each point \( z \) in \( S \).
2. Each reversible entire function is a polynomial.

THEOREM 46

If each of \( u \) and \( v \) is a point-function which is analytic in the region \( R \) and \( K \) is a closed path in \( R \) such that \( |u(z) - v(z)| < |v(z)| \) for each \( z \) in \( K' \), \( \int \frac{u'}{u} = \int \frac{v'}{v} \).

**Hint:** if \( h = \frac{u-v}{v} \) then \( h \) is analytic in a region including \( K' \) and \( h/K \) is a path in \( U \).

THEOREM 47

If the point function \( f \) is analytic in \( R \) and \( w \) is a point in \( R \), the following two statements are equivalent:

1. \( f'(w) = 0 \).
2. If \( r \) is a positive number then there exist points \( p \) and \( q \) such that \( |p-w| \leq r \) and \( |q-w| \leq r \) and \( f(p) = f(q) \).

**Hint:** if the positive number \( r \) has the property that if \( p \) and \( q \) are points and \( |p-w| \leq r \) and \( |q-w| \leq r \) then \( p \) and \( q \) are in \( R \) and \( f(p) \neq f(q) \), then there is a point \( p \) in \( R \) such that \( 0 < |p-w| < r \) and if \( z \) is a point such that \( |z-w| = r \) then

\[
\left| \frac{f(z)-f(p)}{z-p} - \frac{f(z)-f(w)}{z-w} \right| < \left| \frac{f(z)-f(w)}{z-w} \right|.
\]

THEOREM 48

Suppose \( g \) is an analytic function with initial set \( R \), and if \( c > 0 \) and \( S \) is a closed and bounded set lying in \( R \) then there is a reversible point-function \( f \), analytic in \( R \), such that if \( z \) is in \( S \) then \( |f(z)-g(z)| < c \). Either \( g \) is constant or \( g \) is reversible.

**Hint:** if \( p \) and \( q \) are points in \( R \) such that \( g(p) = g(q) \) and \( g \) is not constant, there is a positive number \( r \) such that if \( 0 < |z-p| \leq r \) then \( z \) is in \( R \) and \( g(z) \neq g(p) \).
-- let \( f \) be a reversible point-function, analytic in \( R \), such that if \( z \) is a point and 
\[ |z-p| = r \] 
\[ |[f(z)-f(p)]- [g(z)-g(p)]| < |g(z)-g(p)|. \]

**THEOREM 49**

If \( f \) is a reversible analytic function then \( f^{-1} \) is analytic.

**LEMMA**

If \( w \) is in the initial set \( R \) of \( f \) and \( r \) is a positive number such that \( D_{r}(w) \) lies in \( R \),
(1) \( f(D_{r}(w)) \) is a component of the complement of \( f(C_{r}(w')) \), and
(2) if \( z \) is in \( f(D_{r}(w)) \) then 
\[ f^{-1}(z) = \frac{1}{2\pi i} \int_{C_{r}(w)} \frac{1}{f(w) - z} \, dw. \]

**THEOREM 50**

If the point-function \( h \) is analytic in the simple region \( R \) and the contraction of \( h \) to \( R \) is reversible then \( h(R) \) is simple: moreover, for each closed path \( K \) in \( R \),
(1) if \( P \) is a point not in \( h(R) \) then 
\[ W(h/K, P) = 0, \] and
(2) if \( w \) is in \( R-K' \) then 
\[ W(h/K, h(w)) = W(K, w). \]

**THEOREM 51**

Suppose \( A \) is a point in the region \( R \), \( B \) is a point not in \( R \), and either \( R \) is simple or \( B \) belongs to an unbounded component of the complement of \( R \).
(1) There is a reversible analytic function \( g \), with initial set \( R \), such that if \( K \) is a path from \( A \) to \( z \) in \( R \) then 
\[ g(z) = (A-B) \int_{K} \frac{1}{I-B} \, dw. \]
(2) There is a reversible analytic function \( h \), with initial set \( R \) and slope \( 1 \) at \( \{A, 0\} \), such that \( h(R) \) is bounded.

**THEOREM 52**

Suppose \( A \) is a point in the region \( R \), \( h \) is a reversible analytic function with initial set \( R \) and slope \( 1 \) at \( \{A, 0\} \), and \( B \) is a positive number such that 
\[ h(R) = D_{b}(O). \] If \( k \) is an analytic function with initial set \( R \) and slope \( 1 \) at \( \{A, 0\} \) and 
\[ |k|_{R} \leq b, \] \( k \) is \( h \).

**LEMMA**

If \( g \) is an analytic function with slope \( 1 \) at \( \{0, 0\} \) and 
\[ |g(z)| \leq b \] for each point \( z \) such that 
\[ |z| < b \] then \( g \) is a contraction of the identity function.

**THEOREM 53**

Suppose \( A \) is a point in the region \( R \), \( h \) is a reversible analytic function with initial
set R and slope 1 at \{A,0\} such that \( h(R) \) is bounded, and \( B \) is a point not in \( h(R) \) such that \( |B| < |h|_R \). If \( b > 0 \) and \( b^2 = |B|/|h|_R \), and either R is simple or B belongs to the unbounded component of the complement of h(R), then

1. there is a reversible analytic function \( g \) with initial set \( R \) such that if \( K \) is a path from \( A \) to \( z \) in \( R \) then \( g(z) = b E \left( \frac{1}{2} \int \left( \frac{1}{K - h - B} \right)^4 \right) \),

2. \( g \) has slope \( b \frac{b^4 - 1}{2b} \) at \( \{ A, b \} \) and \( |g|_R = 1 \), and

3. \( \frac{2b}{b + b^2} \frac{g - b}{bg - 1} \) is a reversible analytic function \( f \) with initial set \( R \) and slope 1 at \( \{ A, 0 \} \) such that \( |f|_R = \frac{2b}{1 + b^2} |h|_R < |h|_R \).

**THEOREM 54**

Suppose \( R \) is a region, \( G_R \) is the set to which \( S \) belongs only in case \( S \) is a closed and bounded set lying in \( R \), \( A \) is a function from \( G_R \) to a set of positive numbers, and \( f \) is a sequence each value of which is a point-function analytic in \( R \) such that, if \( n \) is a nonnegative integer and \( S \) is in \( G_R \), \( |f_n|_S \leq A(S) \).

1. There is a function \( B \) from \( G_R \) to a set of positive numbers such that, if \( n \) is a nonnegative integer and \( S \) is in \( G_R \), \( |f_n|_S \leq B(S) \).

2. If \( S \) is in \( G_R \) and \( r > 0 \) and \( S+r \) lies in \( R \) then, if \( n \) is a nonnegative integer and each of \( w \) and \( z \) is a point in \( S \), \( |f_n(w) - f_n(z)| \leq |w-z| B(S+r) \).

3. There is a point-sequence \( t \), with initial set \( T \) lying in \( R \), such that if \( z \) is in \( R \) and \( b > 0 \) then there is a nonnegative integer \( m \) such that \( |z - t_m| < b \).

4. There is a subsequence of \( f \) which converges on \( T \).

5. If \( S \) is in \( G_R \) and \( b > 0 \), there is a positive integer \( n \) such that if \( z \) is in \( S \) then there is an integer \( p \) in \( [0, n] \) such that \( |z - t_p| \leq b \).

6. If \( f \) converges on \( T \) then \( f \) converges uniformly on each member of \( G_R \).

**EXERCISE**

If \( R \) is a region and \( f \) is a sequence each value of which is a point-function with initial set including \( R \), the following two statements are equivalent:

1. There is a continuous point-function \( g \) with initial set \( R \) such that \( f \) has the limit \( g \) uniformly on each closed and bounded set lying in \( R \).

2. \( f \) is continuously convergent on \( R \) -- in the sense that if \( z \) is a sequence with final set lying in \( R \) and with a limit in \( R \) then the sequence \( w \), such that if \( n \) is a nonnegative integer then \( w_n = f_n(z_n) \), converges.
THEOREM 55

Suppose \( A \) is a point in the region \( R \) and \( B \) is a point not in \( R \), either \( R \) is simple or \( B \) belongs to an unbounded component of the complement of \( R \), and \( b \) is the greatest number \( t \) such that if \( h \) is a reversible analytic function from \( R \) onto a bounded region with slope 1 at \( \{A,0\} \) then \( t \leq |h|_R \). The number \( b \) is positive and there is a reversible analytic function \( h \) from \( R \) onto a bounded region such that

1. \( h \) has slope 1 at \( \{A,0\} \) and \( |h|_R = b \),
2. the unbounded component of the complement of \( h(R) \) is the complement of \( D_b(0) \), and
3. if \( R \) is simple then \( h(R) = D_b(0) \).

REMARK

The number \( b \) which is described in the statement of Theorem 55 is sometimes called the analytic radius of \( R \) with respect to \( A \).

EXERCISES

1. If \( w \) is a point and \( r > 0 \) then \( r \) is the analytic radius of \( D_r(w) \) with respect to \( w \), and the contraction of \( I - w \) to \( D_r(w) \) is that reversible analytic function \( h \) with initial set \( D_r(w) \) and slope 1 at \( \{w,0\} \) such that \( h(D_r(w)) \) is a disc with center 0.

2. If \( k \) is a point in \( U \) then \( 1 - |k|^2 \) is the analytic radius of \( U \) with respect to \( k \), and the contraction of \( (1 - |k|^2) \frac{I - k}{1 - k*1} \) to \( U \) is that reversible analytic function \( h \) with initial set \( U \) and slope 1 at \( \{k,0\} \) such that \( h(U) \) is a disc with center 0.

3. If \( A \) is a point in the region \( R \) and \( f \) is a reversible analytic function from \( R \) onto \( U \) which has positive slope at \( \{A,0\} \), then \( \frac{1}{f'(A)} \) is the analytic radius of \( R \) with respect to \( A \).

4. 2 is the analytic radius of the right half-plane with respect to 1, and 4 is the analytic radius of the final set of \( L \) with respect to 0.

THEOREM 56

If \( A_1 \) is a point in the simple region \( R_1 \), \( A_2 \) is a point in the simple region \( R_2 \), and neither \( R_1 \) nor \( R_2 \) is the number-plane, then there is only one reversible analytic function from \( R_1 \) onto \( R_2 \) which has positive slope at \( \{A_1,A_2\} \).

EXERCISES

1. If \( A \) is a point in the region \( R \) and \( h \) is a reversible analytic function from \( R \) onto \( U \) and \( f = \frac{|h'(A)|}{h'(A)} \frac{h - h(A)}{1 - h(A)} \), then \( f \) is the reversible analytic function from \( R \) onto \( U \) which has positive slope at \( \{A,0\} \).
(2) If \( f \) is a reversible analytic function from \( U \) onto \( U \) then there exist a point \( v \) with modulus 1 and a point \( k \) in \( U \) such that \( f \) is a contraction of \( v^* \frac{k-I}{1-k*1} \).

(3) Suppose \( f \) is an analytic function from \( U \) into \( U \) to which \( \{0,0\} \) belongs. If \( g \) is \( \frac{f}{1} \) together with \( \{0,f'(0)\} \) then \( g \) is analytic in \( U \), \( |g(0)| \leq 1 \), and \( g(U) \) is a subset of \( U \) unless \( g \) is constant.

**THEOREM 57 (Convergence Continuation Theorem)**

If, with the suppositions of Theorem 54, the sequence \( f \) converges at each point of a set which has a limit-point in \( R \), then \( f \) is continuously convergent on \( R \).

### PROJECTS

(1) Let \( k \) be an infinite point-sequence with final set lying in \( \bar{U} \) such that if \( n \) is a nonnegative integer and \( |k_n| = 1 \) then \( k_{n+p} = 0 \) for each positive integer \( p \); if \( S \) and \( t \) are sequences such that \( s_0 = t_0 \) and if \( n \) is a nonnegative integer then

a) \( t_n \) is a function from \( U \) such that, if \( z \) is in \( U \), \( t_n(z) = \frac{1-|k_n|^2}{k^n + \frac{1}{k^n}} \) together with \( \{0,k_n\} \)

and b) \( s_{n+1} \) is a function from \( U \) such that, if \( z \) is in \( U \), \( s_{n+1}(z) = s_n(z)[t_{n+1}(z)] \), then the relation \( f \), to which \( \{z,w\} \) belongs only in case \( z \) is in \( U \) and \( w \) is a point such that if \( n \) is a nonnegative integer then \( w \) belongs to \( s_n(\bar{U}) \), is an analytic function from \( U \) into \( \bar{U} \). Conversely, if \( f \) is an analytic function from \( U \) into \( \bar{U} \) then there is only one \( \cdots \).

(2) Suppose \( b \) is a number and \( q \) is a nondecreasing function from \([0,\pi]\) to a set of numbers such that \( q(\pi) - q(-\pi) > 0 \); the function \( f \), to which \( \{z,w\} \) belongs only in case \( z \) is in \( U \) and \( w = \int_{-\pi}^{\pi} \frac{E[i|x|+z]}{E[i|x|-z]} dq + ib \), is an analytic function from \( U \) into the right half-plane. Conversely, if \( f \) is an analytic function from \( U \) into the right half-plane then there is \( \cdots \).
DEFINITION

The number-sphere is the set to which P belongs only in case P is an ordered triple \( \{u,v,w\} \) each term of which is a number and \( u^2 + v^2 + w^2 = 1 \). The ordered triple \( \{0,0,1\} \) is called the point at infinity and is denoted by the symbol \( \infty \). The extended number-plane, denoted by ENP, is the set to which Q belongs only in case either Q is a complex number or \( Q = \infty \).

EXERCISE

Let T be the relation to which \( \{x,y\} \) belongs only in case either \( x = \infty \) and \( y = \infty \) or \( x \) is a complex number and \( y \) is the ordered triple \( \{ \frac{2 \Re x}{|x|^2 + 1}, \frac{2 \Im x}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1} \} \):

1. T is a reversible function from ENP onto the number-sphere, and if \( \{u,v,w\} \) is in the number-sphere and not \( \infty \) then \( T^{-1}\{u,v,w\} = \frac{u+iv}{1-w} \) and \( |T^{-1}\{u,v,w\}|^2 = \frac{1+w}{1-w} \).

2. If \( z \) is a complex number and \( T(z) = \{u,v,w\} \) then \( 4 \frac{1}{|z|^2 + 1} = (u - 0)^2 + (v - 0)^2 + (w - 1)^2 \).

3. If \( x \) and \( y \) are complex numbers and \( T(x) = \{u,v,w\} \) and \( T(y) = \{r,s,t\} \) then \( 4 \frac{|x-y|^2}{(|x|^2 + 1)(|y|^2 + 1)} = (u - r)^2 + (v - s)^2 + (w - t)^2 \).

DEFINITION

The statement that \( t \) is a linear-fractional transformation means that \( t \) is a subset of ENP \( \times \) ENP and there is an ordered quadruple \( \{a,b,c,d\} \) such that

1. Each of \( a, b, c, \) and \( d \) is a point and \( ad - bc \neq 0 \),
2. If \( c = 0 \) then \( V \) is in \( t \) only in case either \( V \) is in \( \frac{aI+b}{d} \) or \( V = \{\infty, \infty\} \), and
3. If \( c \neq 0 \) then \( V \) belongs to \( t \) only in case either \( V \) belongs to \( \frac{aI+b}{cI+d} \) or \( V = \{-\frac{d}{c}, \infty\} \) or \( V = \{\infty, \frac{a}{c} \} \).

REMARK

If each of \( a, b, c, \) and \( d \) is a point and \( ad - bc \neq 0 \) then there is only one linear-fractional transformation of which the point-function \( \frac{aI+b}{cI+d} \) is a subset. Hence, it is customarily agreed that if \( x \) is a point then

1. \( x + \infty = \infty \) since \( \{\infty, \infty\} \) belongs to the linear-fractional transformation which includes the point-function \( x+I \).
2. If \( x \neq 0 \) then \( x = \infty \) since \( \{\infty, \infty\} \) belongs to the linear-fractional transformation which includes the point-function \( xI \).
3. If \( x \neq 0 \) then \( \frac{x}{0} = \infty \) and \( \frac{x}{\infty} = 0 \) since both \( \{0, \infty\} \) and \( \{\infty, 0\} \) belong to the linear-fractional transformation which includes the point-function \( \frac{x}{I} \).
EXERCISES

(1) If each of $t_1$ and $t_2$ is a linear-fractional transformation then $t_1[t_2]$ is a linear-fractional transformation.

(2) If $t$ is a linear-fractional transformation then $t$ is a reversible function from ENP onto ENP, and $t^{-1}$ is a linear-fractional transformation.

(3) If $\{\omega, \omega\}$ belongs to the linear-fractional transformation $t$ then there are points $A$ and $B$ such that $t$ includes $(1-t)A + tB$.

(4) If $\{\omega, \omega\}$ does not belong to the linear-fractional transformation $t$ then there is an ordered point-triple $\{A, B, C\}$ such that $B \neq 0$ and $t$ contains every ordered point-pair $(x, y)$ such that $(y - C)(x - A) = B$.

THEOREM 58

If $a$, $b$, and $c$ are three members of ENP and $A$, $B$, and $C$ are three members of ENP then there is only one linear-fractional transformation to which all of $\{a, A\}$, $\{b, B\}$, and $\{c, C\}$ belong.

Hint: consider the linear-fractional transformation which includes $\frac{1-a}{b-a} \frac{b-c}{1-c}$.

EXERCISE

If $t$ is the linear-fractional transformation to which all of $\{-1, -1\}$, $\{0, i\}$, and $\{1, 1\}$ belong then $\{\omega, -i\}$ belongs to $t$, the $t$-image of the lower half-plane is the unit-disc, and the $t$-image of the real line is the set to which $w$ belongs only in case $w$ is a point different from $-i$ and $|w| = 1$.

DEFINITION

The extended real line is the set to which $Q$ belongs only in case either $Q$ is a number or $Q = \infty$. The statement that $K$ is a circle (in the extended number-plane) means that there is a linear-fractional transformation $t$ such that $K$ is the $t$-image of the extended real line. The symbol $J$ denotes the subset of ENP $\neq$ ENP to which $\{z, w\}$ belongs only in case either $\{z, w\} = \{\omega, \omega\}$ or $z$ is a point and $w = z^*$.

EXERCISES

(1) If $t$ is a linear-fractional transformation then so is $J[t(J)]$.

(2) If $t$ is a linear-fractional transformation then, in order that the $t$-image of the extended real line be the extended real line, it is necessary and sufficient that $J[t] = t[J]$.

(3) If $A$, $B$, and $C$ are three members of ENP then there is only one circle to which all of $A$, $B$, and $C$ belong.
(4) If $t_1$ and $t_2$ are linear-fractional transformations and the $t_1$-image of the extended real line is the $t_2$ image of the extended real line then $t_1[J[t_1^{-1}]] = t_2[J[t_2^{-1}]]$.

**DEFINITION**

If $K$ is a circle, the inversion of ENP in (or with respect to) $K$ is a function $T$ such that if $t$ is a linear-fractional transformation and $K$ is the $t$-image of the extended real line then $T$ is $t[J[t^{-1}]]$.

**EXERCISES**

1. If $T$ is the inversion of ENP in the circle $K$ then $P$ belongs to $K$ only in case $\{P, P\}$ belongs to $T$.
2. $J$ is the inversion of ENP in the extended real line.
3. If each of $T_1$ and $T_2$ is the inversion of ENP in a circle then $T_1[T_2]$ is a linear-fractional transformation.
4. If $T$ is the inversion of ENP in the circle $K$ and $t$ is a linear-fractional transformation then $t[T[t^{-1}]]$ is the inversion of ENP in the circle $t(K)$.

**THEOREM 59**

If $t$ is a linear-fractional transformation, there is a sequence $\{T_p\}_0^7$ each value of which is an inversion of ENP in a circle such that $t = T_1[T_2[T_3[T_4[T_5[T_6[T_7[T_0]]]]]]]$. 

**LEMMA 1**

If $r > 0$, $T_0$ is the inversion of ENP in the circle to which 1, $i$, and $-1$ belong, and $T_1$ is the inversion of ENP in the circle to which $r$, $ir$, and $-r$ belong, then $T_1[T_0]$ is the linear-fractional transformation which includes $r^2I$.

**LEMMA 2**

If $h$ is a point different from 0, $T_0$ is the inversion of ENP in the circle to which 0, $ih$, and $\infty$ belong, and $T_1$ is the inversion of ENP in the circle to which $h$, $h+ih$, and $\infty$ belong, then $T_1[T_0]$ is the linear-fractional transformation which includes $1 + 2h$.

**LEMMA 3**

If $u$ is a point with modulus 1 and $T$ is the inversion of ENP in the circle to which 0, $u$, and $\infty$ belong, $T[J]$ is the linear-fractional transformation which includes $u^2I$.

**LEMMA 4**

There is an inversion $T$ of ENP in a circle such that $J[T]$ includes $\frac{1}{I}$. 
DEFINITION

The chordal metric is a function from ENP \( \times \) ENP, with value at \( \{x,y\} \) denoted by the symbol \( \text{chd}\{x,y\} \) and called the chordal distance from \( x \) to \( y \), such that if each of \( x \) and \( y \) is a complex number then

(1) \( \text{chd}\{x,y\} = 2 |x-y| \left( |1+xy|^2 + |x-y|^2 \right)^{-1/2} \),

(2) \( \text{chd}\{x,\infty\} = \text{chd}\{\infty,x\} = 2 \left( 1 + |x|^2 \right)^{-1/2} \), and

(3) \( \text{chd}\{\infty,\infty\} = 0 \).

REMARK

The chordal metric has the following properties (see first Exercise, this Chapter):

(1) \( 0 \leq \text{chd}\{x,y\} \leq 2 \), \( \text{chd}\{x,y\} = 0 \) only in case \( x = y \), \( \text{chd}\{x,y\} = 2 \) only in case \( \{x,y\} = \{0,\infty\} \) or \( \{x,y\} = \{\infty,0\} \) or \( y = \frac{1}{x^*} \).

(2) \( \text{chd}\{x,y\} = \text{chd}\{y,x\} \) and \( \text{chd}\{x,z\} \leq \text{chd}\{x,y\} + \text{chd}\{y,z\} \).

(3) if \( x \neq 0 \) and \( x \neq \infty \) then \( \text{chd}\{x,\infty\} = \text{chd}\{\frac{1}{x},0\} \).

THEOREM 60

If \( t \) is a linear-fractional transformation then \( t \) is continuous with respect to the chordal metric -- in the sense that if \( \{p,q\} \) is in \( t \) then, for each positive number \( c \), there is a positive number \( b \) such that if \( \{x,y\} \) belongs to \( t \) and \( \text{chd}\{x,p\} < b \) then \( \text{chd}\{y,q\} < c \).

THEOREM 61

If \( t \) is a linear-fractional transformation, in order that \( \text{chd}\{t(x),t(y)\} = \text{chd}\{x,y\} \) for each \( x \) and \( y \) in ENP, it is necessary and sufficient that there be a point \( k \) and a point \( v \) with modulus 1 such that \( t \) contains each \( \{z,w\} \) such that \( \frac{k-w}{1+k\bar{w}} = v \frac{k-z}{1+k\bar{z}} \).

DEFINITION

The point-function \( f \) has order \( j \) at \( \infty \) provided \( f[\frac{1}{t}] \) has order \( j \) at 0. If \( R > 0 \) and the point-function \( f \) is analytic at each point with modulus greater than \( R \) then \( \infty \) is a zero, a pole, or an essential singularity of \( f \) accordingly as 0 is a zero, a pole, or an essential singularity of \( f[\frac{1}{t}] \).

DEFINITION

The statement that \( F \) is a meromorphic function means that \( F \) is a function from a subset of ENP into ENP such that

(1) either the initial set of \( F \) is ENP or there is a linear-fractional transformation \( t \) such that the initial set of \( F \) is the \( t \)-image of a region,
and (2) either $F$ is constant or there is an analytic function $g$ included in $F$ such that if \{p,q\} is in $F$ then, for each positive number $c$, there is a positive number $b$ such that if $x$ is in $ENP$ and $0 < chd\{x,p\} < b$ then $chd\{g(x),q\} < c$.

**THEOREM 62**

If $F$ is a nonconstant meromorphic function with initial set $ENP$, there exist polynomials $P$ and $Q$ such that $F$ is the meromorphic function with initial set $ENP$ which includes $\frac{P}{Q}$.

**PROBLEM**

Is it true that each reversible meromorphic function with initial set $ENP$ is a linear-fractional transformation?
DEFINITION

An analytic relation is a relation $F$, each member of which belongs to an analytic function included in $F$, such that if $g$ and $h$ are analytic functions included in $F$ then there is an ordered pair $\{x,y\}$ such that

(1) each of $x$ and $y$ is a continuous function from $[0,1]$ to the number-plane and $x$ is not constant on any subinterval of $[0,1]$,

(2) there is a positive number $b$ less than 1 such that if $s$ is in $[0,b]$ and $t$ is in $[1-b,1]$ then $\{x(s),y(s)\}$ belongs to $g$ and $\{x(t),y(t)\}$ belongs to $h$,

and (3) if $0 < u < 1$ then there is a positive number $c$ and an analytic function $k$ included in $F$ such that if $v$ is in $[u-c,u+c]$ then $\{x(v),y(v)\}$ belongs to $k$.

EXERCISES

(1) Every analytic function is an analytic relation.

(2) If $F$ and $G$ are analytic relations, $H$ is the sum of $F$ and $G$, and there is an analytic function which is included both in $F$ and in $G$, then $H$ is an analytic relation.

(3) If $F$ is an analytic relation then there is an analytic relation which includes $F$ and is not included in any other analytic relation.

(4) Is it true that if $f$ is an analytic function then there is an analytic function which includes $f$ and is not included in any other analytic function.

THEOREM 63

If $g$ and $h$ are analytic functions included in the analytic relation $F$ and $\{A_1,A_2\}$ is in $g$ and $\{B_1,B_2\}$ is in $h$, then there is a nonnegative integer $n$ and a sequence $\{f_p\}_{p=0}^{n+1}$ each value of which is an analytic function included in $F$ such that

(1) $f_0$ is a subset of $g$ to which $\{A_1,A_2\}$ belongs and $f_{n+1}$ is a subset of $h$ to which $\{B_1,B_2\}$ belongs, and

(2) if $p$ is a nonnegative integer less than $n+1$ then the sum of $f_p$ and $f_{p+1}$ is an analytic function.

EXERCISE

If $n$ is a nonnegative integer and $\{f_p\}_{p=0}^{n+1}$ is a sequence, each value of which is an analytic function, such that if $p$ is a nonnegative integer less than $n+1$ then the sum of $f_p$ and $f_{p+1}$ is an analytic function, then, for each $A$ in the initial set of $f_0$ and each $B$ in the initial set of $f_{n+1}$, there exist sequences $\{z_p\}_{p=0}^{n+2}$ and $\{M_p\}_{p=0}^{n+1}$ such that

(1) $z_0$ is $A$, $z_{n+2}$ is $B$, and if $p$ is a nonnegative integer less than $n+1$ then $z_{p+1}$ belongs to the initial set of $f_p$ and to the initial set of $f_{p+1}$, and

(2) if $p$ is an integer in $[0,n+1]$ then $M_p$ is a path from $z_p$ to $z_{p+1}$ in the initial set of $f_p$.
DEFINITION

If \( g \) is an analytic function and \( K \) is a path, the statement that \( F \) is an analytic continuation of \( g \) along \( K \) means that \( F \) is an analytic relation including \( g, K \) is a path from a point \( A \) in the initial set of \( g \) to a point \( B \), and there exist a nonnegative integer \( n \) and a sequence \( \{f_p\}_{n+1} \), each value of which is an analytic function included in \( F \), and sequences \( \{z_p\}_{n+2}^0 \) and \( \{M_p\}_{n+1}^0 \) such that

1. \( f_0 \) is a subset of \( g \) and if \( p \) is a nonnegative integer less than \( n+1 \) then the sum of \( f_p \) and \( f_{p+1} \) is an analytic function,

2. \( z_0 \) is \( A \) and is in the initial set of \( f_0 \), \( z_{n+2} \) is \( B \) and is in the initial set of \( f_{n+1} \), and if \( p \) is a nonnegative integer less than \( n+1 \) then \( z_{p+1} \) belongs to the initial set of \( f_p \) and to the initial set of \( f_{p+1} \),

3. if \( p \) is an integer in \([0,n+1]\) then \( M_p \) is a path from \( z_p \) to \( z_{p+1} \) in the initial set of \( f_p \), and

4. \( K = \sum_{p=0}^{n+1} M_p (+) \).

EXERCISE

If \( \{A_1, A_2\} \) is a member of the analytic function \( g \) then, in order that \( \{B_1, B_2\} \) should belong to an analytic relation including \( g \), it is necessary and sufficient that \( \{B_1, B_2\} \) should belong to an analytic continuation of \( g \) along a path from \( A_1 \) to \( B_1 \).

THEOREM 64 (Monodromy Theorem)

If \( R \) is a simple region which includes the initial set of the analytic function \( g \), \( A \) is a point in the initial set of \( g \), and for each point \( B \) in \( R \) and each path \( K \) from \( A \) to \( B \) in \( R \) there is an analytic continuation of \( g \) along \( K \), then \( g \) is included in an analytic function with initial set \( R \).

Hint: show that, by Theorem 55, it suffices to consider the case that \( A \) is \( O \) and \( R \) is either the number-plane or a disc with center \( O \) -- and, in this case, it suffices to consider continuations of \( g \) along linear paths from \( O \) to points of \( R \).

DEFINITION

If \( F \) is an analytic relation, the statement that \( z \) is an \( F \)-boundary-point of \( R \) means

1. \( R \) is the initial set of an analytic function \( g \) which is included in \( F \) and \( z \) belongs to the boundary of \( R \), and

2. there do not exist two analytic functions \( h_1 \) and \( h_2 \), each of which is a subset of \( F \), such that \( h_1 \) is a subset of \( g \) and of \( h_2 \), \( z \) belongs to the boundary of the initial set of \( h_1 \), and \( z \) belongs to the initial set of \( h_2 \).
THEOREM 65

If the power-series \( \sum A_p (I-w)^p \) about \( w \) has the radius of convergence \( r \) and has the limit \( g \) on the disc \( D_r(w) \), and \( F \) is the analytic relation which includes \( g \) and is not included in any other analytic relation, then there is a boundary-point of \( D_r(w) \) which is an \( F \)-boundary-point of \( D_r(w) \).

DEFINITION

If \( F \) is an analytic relation, the derivative of \( F \), denoted by \( F' \), is the set to which \( V \) belongs only in case there is an analytic function \( g \) included in \( F \) such that \( V \) belongs to \( g' \).

EXERCISE

If \( F \) is an analytic relation then \( F' \) is an analytic relation.

THEOREM 66

Every analytic relation in the derivative of an analytic relation.

REMARK

Compare this result with Theorem 15, and with the Corollary to Theorem 16.

THEOREM 67

Suppose \( R \) is a region, each of \( g \) and \( h \) is a point-function analytic in \( R \), and \( O \) does not belong to \( g'(R) \). If \( F \) is the set to which \( V \) belongs only in case there is a point \( z \) in \( R \) such that \( V = \{ g(z), h(z) \} \), then \( F \) is an analytic relation.

DEFINITION

A branch-point of the analytic relation \( F \) is a point \( B \) such that there exist three analytic functions \( g_1 \), \( g_2 \), and \( g_3 \), all included in \( F \), with respective initial sets \( R_1 \), \( R_2 \), and \( R_3 \), such that

1. \( B \) is a boundary-point of each of \( R_1 \), \( R_2 \), and \( R_3 \),
2. there is a positive number \( r \) such that if \( z \) is a point and \( 0 < |z-B| < r \) then \( z \) belongs either to \( R_1 \) or to \( R_2 \) or to \( R_3 \),
3. the sum of \( g_1 \) and \( g_2 \) is an analytic function and the sum of \( g_2 \) and \( g_3 \) is an analytic function, and
4. the sum of \( g_3 \) and \( g_1 \) is not a function.
THEOREM 68

Suppose \( f \) is a nonconstant analytic function, \( \{A,B\} \) is a member of \( f \) at which \( f \) has slope 0, and \( F \) is the inverse of the subset of \( f \) to which \( V \) belongs only in case 0 is not the slope of \( f \) and \( V \). The set \( F \) is an analytic relation, and \( B \) is a branch-point of \( F \).

**LEMMA 1**

There exist an integer \( n \) greater than 1 and an analytic function \( g \) such that \( g(A) \neq 0 \) and \( f = B + (I-A)^n g \).

**LEMMA 2**

There exist a positive number \( s \) and a reversible analytic function \( h \) such that \( h(B) = A \) and if \( z \) is a point in \( D_s(B) \) then \( f(h(z)) = B + (z - B)^n \).

**Hint:** The point \( A \) belongs to a convex region \( D \) such that \( 0 \) does not belong to \( g(D) \) -- let \( b \) be a point such that \( b^n = g(A) \) and consider a point-function \( k \) such that if \( z \) is in \( D \) then \( k(z) = B + (z-A) \left( \frac{1}{n} \int_{[A;z]} k' \right) \).

**LEMMA 3**

Let \( r \) be a positive number such that \( 2 r^2 < s^2 \), \( R_1 \) be the set to which \( z \) belongs only in case \( 0 < |z-B| < s \) and \( \text{Re}(z-B) > 0 \) or \( \text{Im}(z-B) > 0 \), \( R_2 \) be the set to which \( z \) belongs only in case \( 0 < |z-B| < s \) and \( \text{Im}(z-B) > 0 \) or \( \text{Re}(z-B) < 0 \), and \( R_3 \) be the set to which \( z \) belongs only in case \( 0 < |z-B| < s \) and either \( \text{Re}(z-B) < 0 \) or \( \text{Im}(z-B) < 0 \). There exist reversible analytic functions \( \varnothing_1, \varnothing_2, \) and \( \varnothing_3 \), with respective initial sets \( R_1, R_2, \) and \( R_3 \), such that

1. if \( z \) is in \( R_1 \) then \( \varnothing_1(z) = B + r^{1/n} \left( \frac{1}{n} \int_{[B+r;z]} \frac{1}{I-B} \right) \),

2. the sum of \( \varnothing_1 \) and \( \varnothing_2 \) is an analytic function and the sum of \( \varnothing_2 \) and \( \varnothing_3 \) is an analytic function, and

3. the sum of \( \varnothing_3 \) and \( \varnothing_1 \) is not a function.

**EXERCISES**

1. If \( F \) is the inverse of the exponential function \( E \) then \( F \) is an analytic relation (the natural logarithmic relation), and \( O \) is a branch-point of \( F \).

2. Suppose \( A, B, \) and \( C \) are points such that \( A \neq C \), \( f = B + (I-A)^2 (I-C) \), and \( F \) is the inverse of the subset of \( f \) to which \( V \) belongs only in case \( f \) has slope different from 0 at \( V \). The point \( B \) is a branch-point of \( F \) belonging to the initial set of \( F \).

3. Use Theorem 67 to show that the set \( F \), to which \( V \) belongs only in case \( V \) is
an ordered pair \( \{x,y\} \) such that \( x \) is a point and \( y \) is a point different from 0 and 
\[ x^2 + y^2 = 1, \]
is an analytic relation.

**DEFINITION**

The point-relation \( F \) is **algebraic** provided there is a positive integer \( n \) and a **complex matrix** \( A \), i.e., a function \( A \) from the ordered integer-pairs in \([0,n] \times [0,n]\) to the complex numbers, such that for some integer \( p \) in \([0,n]\) the **row** \( A[p,1] \) of \( A \) is not the constant 0 on the integers in \([0,n]\), and \( \{x,y\} \) belongs to \( F \) only in case each of \( x \) and \( y \) is a point and 
\[ \sum_{p=0}^{n} \sum_{q=0}^{n} A(p,q) x^p y^q = 0. \]

**EXERCISES**

1. If \( F \) is an algebraic relation then \( F^{-1} \) is an algebraic relation.
2. If the algebraic relation \( F \) includes the analytic function \( g \) then \( F \) includes every analytic relation of which \( g \) is a subset.
3. If the analytic relation \( F \) is included in an algebraic relation then \( F' \) is included in an algebraic relation.
4. Extend the notion of branch-point to give meaning to the statement that \( \infty \) is a branch-point of the analytic relation \( F \).
5. Formulate a definition of **meromorphic relation**.
THEOREM 1.1

If $R$ is a simple region which is not the number-plane then the complement of $R$ does not have a bounded component.

SUGGESTION

Let $h$ be a reversible analytic function from the unit-disc $U$ onto the region $R$.

Lemma 1. If $t$ is a number in $(0,1)$, the complement of $h(C_t(O))'$ has only two components and the bounded component is the $h$-image of the bounded component of the complement of $C_t(O)'$.

Lemma 2. If $S_1$ and $S_2$ are mutually exclusive closed and bounded point-sets then there exist mutually exclusive open point-sets $T_1$ and $T_2$ such that $S_1$ lies in $T_1$ and $S_2$ lies in $T_2$.

Lemma 3. If $A$ is a point not in $R$ and $M$ is a positive number then there exist a point $B$ such that $|B-A| = M$ and a connected point-set which contains both $A$ and $B$ and lies in the complement of $R$.

THEOREM 1.2

Suppose $A$ is a point in the region $R$ and $B$ is a point not in $R$ such that if $K$ is a closed path in $R$ then $W(K,B) = 0$:

1. there is a reversible analytic function, with initial set $R$ and bounded final set, which has positive slope at $\{A,0\}$.

2. if $A = 0$ and there is a point $u$ in $R$ such that $|B| < |u|$ then there exist a point $v$ in $R$ and a reversible analytic function $f$, with initial set $R$ and slope 1 at $\{0,0\}$, such that $|f|_R < |v|$.

THEOREM 1.3

If $R$ is a region which is not the number-plane then each two of the following five statements are equivalent:

1. $R$ is simple.

2. There is a reversible analytic function from $R$ onto $U$.

3. $R$ is simply connected -- in the sense that the complement of $R$ does not have a bounded component.

4. If $K$ is a closed path in $R$ then $W(K,P) = 0$ for each point $P$ not in $R$.

5. If each of $A$ and $B$ is a point in $R$ and $K_0$ and $K_1$ are paths from $A$ to $B$ in $R$ then $K_1$ is homotopically equivalent to $K_0$ in $R$ -- in the sense that there exist a
positive number $p$ and a function $F$ from $[0,1] \times [0,1]$ into $\mathbb{R}$ such that

i) $F$ is continuous -- in the sense that if $(s, t)$ is in $[0,1] \times [0,1]$ and $c$ is a positive number, there is a positive number $b$ such that if $(u, v)$ is in $[0,1] \times [0,1]$ and $|u-s| < b$ and $|v-t| < b$ then $|F(u,v) - F(s,t)| < c$,

ii) if $t$ is in $[0,1]$ and $x = F[I, t]$ then $x(0) = A$, $x(1) = B$, $\int_0^1 |dx| \leq p$,

and iii) $F[I, 0]$ belongs to $K_0$ and $F[I, 1]$ belongs to $K_1$.

**THEOREM 1.4**

If $A$ is a point in the region $R$, $T_A$ is the relation to which $\{K_0, K_1\}$ belongs only in case each of $K_0$ and $K_1$ is a path from $A$ to $A$ in $R$ and $K_1$ is homotopically equivalent to $K_0$ in $R$, and $G$ is the final set of the relation to which $\{K, S\}$ belongs only in case $K$ is a path from $A$ to $A$ in $R$ and $S$ is the set to which $K_1$ belongs only in case $\{K, K_1\}$ belongs to $T_A$, then the following statements are true:

1. $T_A$ is an equivalence relation.
2. If each of $\{K_1, K_3\}$ and $\{K_2, K_4\}$ is in $T_A$, $\{K_1, K_2, K_3, K_4\}$ is in $T_A$.
3. There is a function $H$ from $G \times G$ into $G$ such that if $\{S_1, S_2\}$ is in $G \times G$ and $K_1$ is in $S_1$ and $K_2$ is in $S_2$ then $H(S_1, S_2)$ is the member of $G$ containing $K_1$ and $K_2$.
4. The ordered pair $\{G, H\}$ is a group -- in the sense that

   i) if each of $S_1$, $S_2$, and $S_3$ is in $G$ then $H(S_1, H(S_2, S_3)) = H(H(S_1, S_2), S_3)$,

   ii) there is an $S_0$ in $G$ such that if $S$ is in $G$ then $H(S_0, S) = H(S, S_0) = S$,

and iii) if $S_1$ is in $G$ then there is a member $S_2$ of $G$ such that

$$H(S_1, S_2) = H(S_2, S_1) = S_0.$$  

**REMARK**

The group $\{G, H\}$ described in Theorem 1.4 is sometimes called the homotopy-group of the region $R$ (with respect to the point $A$).

**PROBLEM**

If $A_1$ and $A_2$ are points in the region $R$, $\{G_1, H_1\}$ is the homotopy-group of $R$ with respect to $A_1$, and $\{G_2, H_2\}$ is the homotopy-group of $R$ with respect to $A_2$, then $\{G_2, H_2\}$ is isomorphic to $\{G_1, H_1\}$ -- in the sense that there is a reversible function $f$ from $G_1$ onto $G_2$ such that if $\{X, Y\}$ is in $G_1 \times G_1$ then $H_2(f(X), f(Y)) = f(H_1(X, Y))$.

**EXERCISE**

Suppose $R$ is the final set of the exponential function $E$, $\{G_0, H_0\}$ is the homotopy-group of $R$ with respect to $1$, and $G$ is the set to which $S$ belongs only in case there is an integer $n$ such that $S$ is the linear-fractional transformation including $I + 2\pi i$. 
(1) If $s$ is in $G$ then $E[s] = E$.

(2) If each of $s_1$ and $s_2$ is in $G$ then each of $s_1^{-1}$ and $s_1[s_2]$ is in $G$.

(3) If $t$ is a linear-fractional transformation to which $\{\infty, \infty\}$ belongs and $E[t] = E$ then $t$ belongs to $G$.

(4) If each of $z_1$ and $z_2$ is a point and $E(z_1) = E(z_2)$ then there is only one member of $G$ to which $\{z_1, z_2\}$ belongs.

(5) If $K$ is a path from 1 in $R$, there is a path $K_0$ from 0 such that $K = E/K_0$.

(6) If $f$ is the relation to which $\{s, X\}$ belongs only in case $s$ is in $G$ and $X$ is the member of $G_0$ to which $E/\{0; s(0)\}$ belongs, then $f$ is a reversible function from $G$ onto $G_0$ such that if $\{s, t\}$ is in $G \times G$ then $H_0(f(s), f(t)) = f(s[t])$.

**DEFINITION**

The hyperbolic metric is a function from $U \times U$, with value at the member $\{x, y\}$ of $U \times U$ denoted by the symbol $\text{hyp}(x, y)$ and called the hyperbolic distance from $x$ to $y$, such that if each of $x$ and $y$ is a point in $U$ then

$$\text{hyp}(x, y) = \frac{1}{2} L\left( \frac{|1 - x*y|}{|1 - x*y|} + \frac{|y - x|}{|y - x|} \right).$$
THEOREM 2.1

If $A$ is a point in the region $R$ and the complement of $R$ has an unbounded component, then there is only one analytic relation $f$ such that

1. $f$ has initial set $R$ and final set the unit-disc $U$,
2. $f^{-1}$ is an analytic function with positive slope at $\{0, A\}$, and
3. if $K$ is a path from $A$ in $R$, there is a path $K_0$ from $0$ such that $K = f^{-1}/K_0$.

SUGGESTION

Suppose $A$ is a point in the region $R$ and the complement of $R$ has an unbounded component. There is a reversible analytic function, with initial set $R$ and bounded final set, which has slope 1 at $\{A, 0\}$. Let $H$ be the collection to which $h$ belongs only in case

1. $h$ is an analytic relation from $R$ onto a bounded point-set,
2. $h^{-1}$ is an analytic function with slope 1 at $\{0, A\}$, and
3. if $K$ is a path from $A$ in $R$, there is a path $K_0$ from $0$ such that $K = h^{-1}/K_0$.

If $S$ is a subset of the initial set of the point-relation $F$ and $F(S)$ is bounded, let $|F|_S$ denote the least number not less than the modulus of any point in $F(S)$.

Lemma 1. If $h$ is a member of $H$, $\{B, C\}$ belongs to $h$, and $K$ is a path from $B$ in $R$, there is a path $K_0$ from $C$ such that $K = h^{-1}/K_0$.

Lemma 2. If $h$ is a member of $H$, $\{B_1, C_1\}$ belongs to $h$, and $B_2$ is a point in $R$ different from $B_1$, then $\{B_1, C_1\}$ belongs to an analytic function which is a subset of $h$ and has $B_2$ in its initial set.

Lemma 3. If each of $h$ and $k$ belongs to $H$, $h(R)$ is the disc to which $z$ belongs only in case $|z| < |h|_R$, and $|k|_R \leq |h|_R$, then $k$ is $h$.

Lemma 4. If $h$ is a member of $H$ and $h(R)$ is not the disc to which $z$ belongs only in case $|z| < |h|_R$, then there is a member $f$ of $H$ such that $|f|_R \leq |h|_R$.

Lemma 5. If $h$ is the greatest number $t$ such that if $h$ is in $H$ then $t \leq |h|_R$, then $b$ is positive and there is only one member of $H$ with final set $D_b(0)$.

THEOREM 2.2

Suppose $A$ is a point in the region $R$, $f$ is an analytic relation having all the properties (1) and (2) and (3) enumerated in the statement of Theorem 2.1, and $\emptyset = f^{-1}$.

1. If $z$ and $w$ are points in $U$ such that $\emptyset(z) = \emptyset(w)$, $u = \emptyset'(z)/\emptyset'(z)$ and $v = \emptyset'(w)/\emptyset'(w)$, and $t$ is the linear-fractional transformation which contains all ordered point-pairs $\{x, y\}$ such that $u \frac{x - z}{1 - z \# x} = v \frac{y - w}{1 - w \# y}$, $t(U) = U$ and $\emptyset[t] = \emptyset$. 

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(2) If $G$ is the collection to which $s$ belongs only in case $s$ is a linear-fractional transformation such that $s(U) = U$ and $\varnothing[s] = \varnothing$, then $G$ is a transformation-group in the sense that if each of $s_1$ and $s_2$ is in $G$ then each of $s_1^{-1}$ and $s_1[s_2]$ is in $G$.

(3) If $s$ is a member of $G$ and there is a point $x$ in $U$ such that $s(x) = x$ then $s$ is the linear-fractional transformation which includes the identity function $I$.

REMARK

The transformation-group described in Theorem 2.2 (2) is sometimes called the fundamental group of the region $R$ (with respect to the point $A$). If $G$ is a transformation-group and $\varnothing$ is an analytic function such that if $t$ is in $G$ then $\varnothing[t] = \varnothing$, the function $\varnothing$ is said to be automorphic with respect to the group $G$.

THEOREM 2.3

Suppose $t$ is a linear-fractional transformation, $t(U) = U$, and if $x$ is a point in $U$ then $t(x) \neq x$. One of the following two cases occurs:

(1) $t$ contains only one ordered pair of the form $(x,x)$ -- in this case there exist a linear-fractional transformation $s$ and a point $b$ such that $|b - \frac{1}{2}| = \frac{1}{2}$, $b^2 \neq b$, and for each $u$ in $U$ $s^{-1}(t(s(u))) = \frac{1}{1 - b^*} \frac{u - b^*}{1 - bu}$; if $z$ is a sequence such that $z_0$ is in $U$ and, for each nonnegative integer $n$, $z_{n+1} = t(z_n)$ then $z$ is a reversible sequence which has the limit $s(1)$.

(2) $t$ contains only two ordered pairs of the form $(x,x)$ -- in this case there exist a linear-fractional transformation $t$ and a number $c$ such that $0 < c < 1$ and for each $u$ in $U$ $s^{-1}(t(s(u))) = \frac{u + c}{1 + cu}$; if $z$ is a sequence such that $z_0$ is in $U$ and, for each nonnegative integer $n$, $z_{n+1} = t(z_n)$ then $z$ is a reversible sequence with the limit $s(1)$.

THEOREM 2.4

Suppose $A$ is a point in the region $R$, $f$ is an analytic relation having all the properties (1) and (2) and (3) enumerated in the statement of Theorem 2.1, $\varnothing = f^{-1}$, and $h$ is an analytic function with initial set the simple region $D$ and final set lying in $R$. If $\{p,q\}$ belongs to $h$ and $\{z,q\}$ belongs to $\varnothing$ then

(1) there is an analytic function $g$ from $D$ into $U$ such that $\{p,z\}$ belongs to $g$ and $h = \varnothing[g]$, and

(2) if $r > 0$ and $D$ includes $D_r(0)$ then $r |h'(p)| \leq (1 - |z|^2) |\varnothing'(z)|$.

REMARK

In the preceding theorem we see that if $D$ is the number-plane then $h$ is constant; hence, $R$ is not the number-plane and the complement of $R$ contains at least two points.
PROBLEMS

There exists an analytic function $\phi$ with initial set $U$ such that

1. $0$ does not belong to $\phi'(U),$ 
2. $\phi$ has positive slope at $\{0, \phi(0)\},$ and 
3. there is a path $K$ from $\phi(0)$ in $\phi(U)$ such that if $K_0$ is a path from $O$ in $U$ then $K \neq \phi/K_0.$

There exist points $A_1$ and $A_2,$ belonging respectively to regions $R_1$ and $R_2,$ such that $R_1$ is not simple and the fundamental group of $R_1$ with respect to $A_1$ is the fundamental group of $R_2$ with respect to $A_2.$

If $t$ is a linear-fractional transformation and $t(U) = U$ then, for each two points $x$ and $y$ in $U,$ $\text{hyp}(t(x), t(y)) = \text{hyp}(x, y);$ if $f$ is an analytic function from $U$ into $U$ and $f$ is not the contraction to $U$ of a linear-fractional transformation $t$ such that $t(U) = U$ then, for each two points $x$ and $y$ in $U,$ $\text{hyp}(f(x), f(y)) < \text{hyp}(x, y).$
NOTATION

Let $\omega = E(2\pi i/3)$, so that $\omega^* = \omega^2$ and $1 + \omega + \omega^2 = 0$. Let $t$ be the linear fractional transformation which includes $\frac{1 + i\omega}{1 + i\omega^2}$, so that $t$ maps the right half-plane onto $U$ and $t^{-1}$ includes $i\omega^2 \frac{1 - \omega^2}{1 - i}$. Let $C_1$, $C_2$, and $C_3$ be circles such that if $n$ is 1 or 2 or 3 then $z$ belongs to $C_n$ only in case $|z + 2\omega^n|^2 = 3$. Let $T_1$, $T_2$, and $T_3$ be the inversions of ENP in the circles $C_1$, $C_2$, and $C_3$, respectively.

Problem 1. $T_1[t] = t[J]$, $T_2[t] = t[J + 21]$, and $T_3[t] = t[\frac{1/4}{J + 1/2} + \frac{1}{2}]$.

Let $v$ and $w$ be the linear-fractional transformations including $-\frac{2\omega^2 - 1}{2\omega + 1}$ and $\omega I$, resp.

Problem 2. $v[t] = t[I + i]$ and $w[t] = t[i + \frac{1}{I}]$; moreover, $v(C_1) = C_2$, $w(C_1) = C_2$, and $w(C_2) = C_3$, so that $v[T_1] = T_2[v]$, $w[T_1] = T_2[w]$, and $w[T_2] = T_3[w]$.

Let $E_1$ denote the contraction of $-E[-\pi t^{-1}]$ to $U$.

Problem 3. $E_1(U)$ is the point-set to which $z$ belongs only in case $0 < |z| < 1$; $E_1$ maps the common part of $C_1$ and $U$ onto the number-segment $(-1, 0)$, and maps the common part of $C_2$ and $U$ onto the number-segment $(0, 1)$; $E_1[T_1] = E_1[T_2] = J[E_1]$; $E_1[v] = -E_1$.

Let $V$ be a sequence such that $V_1$ is the point-set to which $z$ belongs only in case $|z| < 1$, $|z + 2\omega^2|^2 > 3$, and $|z - 2 \frac{1 - \omega}{3}|^2 > \frac{1}{3}$, and if $n$ is a positive integer then $V_{2n}$ is the sum of $V_{2n-1}$ and $v(V_{2n-1})$, and $V_{2n+1}$ is the sum of $V_{2n}$ and $T_1(V_{2n})$.

Problem 4. If $n$ is a positive integer, $T_1(V_{2n-1}) = V_{2n-1}$ and $z$ is in $t^{-1}(V_{2n-1})$ only in case $\text{Re } z > 0$ and $-n < \text{Im } z < n$.

Problem 5. There is a sequence $g$ such that $g_1$ is the contraction of $E_1$ to $V_1$ and if $n$ is a positive integer then

1. $g_{2n}$ is a function with initial set $V_{2n}$ such that if $z$ is in $V_{2n-1}$ then $g_{2n}(z) = g_{2n-1}(z)$ and $g_{2n}(v(z)) = -g_{2n-1}(z)$, and
2. $g_{2n+1}$ is a function with initial set $V_{2n+1}$ such that if $z$ is in $V_{2n}$ then $g_{2n+1}(z) = g_{2n}(z)$ and $g_{2n+1}(T_1(z)) = J(g_{2n}(z))$.

Problem 6. If $n$ is a positive integer and $z$ is in $v(V_{2n-1})$ then $T_2(z)$ is in $v(V_{2n-1})$ and $g_{2n}(T_2(z)) = J(g_{2n}(z))$; $E_1$ is the set to which $Q$ belongs only in case there is a positive integer $n$ such that $Q$ belongs to $g_n$. 

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Let $W$ be a sequence such that $W_1$ is the point-set to which $z$ belongs only in case
\[ |z| < 1, \quad |z+2\omega^2| > 3, \quad |z+2| > 3, \quad |z - 2 \frac{\omega^2 - \omega}{3}|^2 > \frac{1}{3}, \ \text{and} \quad |z - 2 \frac{1-\omega}{3}|^2 > \frac{1}{3}, \]
and if $n$ is a positive integer then $W_{3n-1}$ is the sum of $W_{3n-2}$ and $w(W_{3n-2})$, $W_{3n}$ is the sum of $W_{3n-1}$ and $w(W_{3n-1})$, and $W_{3n+1}$ is the sum of $W_{3n}$ and $T_1(W_{3n})$.

**Problem 7.** $W_1$ is a simple region and $t^{-1}(W_1)$ is the region to which $z$ belongs only in case $\text{Re} \ z > 0$, $-1 < \text{Im} \ z < 1$, $|z + \frac{i}{2}| > \frac{1}{2}$, and $|z - \frac{i}{2}| > \frac{1}{2}$.

**Problem 8.** If $z$ is in $U$ then there is a positive integer $n$ such that $z$ is in $W_n$.

Let $f_1$ be the reversible analytic function, with initial set $t^{-1}(W_1)$ and final set the right half-plane, which has positive slope at $\{1,1\}$.

**Problem 9.** $f_1[J] = J[f_1]$ and, if $z$ is in $t^{-1}(W_1)$ and $\text{Im} \ z \neq 0$, $[\text{Im} \ z][\text{Im} \ f_1(z)] > 0$; the contraction of $f_1$ to the set of all positive numbers is an increasing function with final set the set of all positive numbers.

Let $W_0$ be the point-set to which $z$ belongs only in case $z$ is in $W_1$ and $|z+2\omega|^2 > 3$, and $f_2$ be a function with initial set $W_0$ such that if $z$ belongs to $W_0$ then
\[ f_2(z) = \frac{f_1(t^{-1}(z))^2 - f_1(\omega/i)^2}{f_1(t^{-1}(z))^2 - f_1(i/\omega)^2}. \]

**Problem 10.** $f_2$ is a reversible analytic function with final set $U$, $f_2[w] = w[f_2]$, and $f_1(\omega/i) = \omega f_1(i/\omega)$.

Let $f_3 = \frac{f_1[f^{-1}]^2}{f_1[f^{-1}]^2 + f_1(\omega/i)f_1(i/\omega)}$.

**Problem 11.** $f_3[T_1] = J[f_3]$; it $z$ belongs to $W_0$ then
\[ f_3(z) = \frac{1 - \omega}{1 - \omega^2} \frac{f_2(z) - \omega^2}{f_2(z) - \omega} \quad \text{and} \quad f_3(\omega z) = \frac{1}{1 - f_3(z)}; \]

$f_3$ is a reversible analytic function and $z$ belongs to the final set of $f_3$ only in case either $\text{Im} \ z \neq 0$ or $0 < \text{Re} \ z < 1$; $f_3$ maps the common part of $C_1$ and $U$ onto $\langle 0, 1 \rangle$.

**Problem 12.** There is a sequence $h$ such that $h_1 = f_3$ and, for each positive integer $n$,

(1) $h_{3n-1}$ is a function with initial set $W_{3n-1}$ such that if $z$ belongs to $W_{3n-1}$ then $h_{3n-1}(z) = h_{3n-2}(z)$ and $h_{3n-1}(\omega z) = \frac{1 - h_{3n-2}(z)}{1 - h_{3n-2}(z)}$.
(2) $h_{3n}$ is a function with initial set $W_{3n}$ such that if $z$ belongs to $W_{3n-1}$ then $h_{3n}(z) = h_{3n-1}(z)$ and $h_{3n}(wz) = \frac{1}{1 - h_{3n-1}(z)}$, and

(3) $h_{3n+1}$ is a function with initial set $W_{3n+1}$ such that if $z$ belongs to $W_{3n}$ then $h_{3n+1}(z) = h_{3n}(z)$ and $h_{3n+1}(1/(z)) = J(h_{3n}(z))$.

**DEFINITION**

The **modular function** $E_2$ is the set to which $Q$ belongs only in case there is a positive integer $n$ such that $Q$ belongs to $h_n$.

**THEOREM 3.1**

$E_2$ is an analytic function with initial set $U$ such that

1. $z$ belongs to $E_2(U)$ only in case $z$ is a point and $z(1-z) \neq 0$,
2. $E_2$ maps the common part of $C_1$ and $U$ onto the segment $(0,1)$,
3. $E_2[T_1] = E_2[T_2] = E_2[T_3] = J(E_2)$ and $E_2[w] = \frac{1}{1 - E_2}$,
4. $E_2(0) = 1 + \omega$ and $E_2^{-1}$ is an analytic relation, and
5. if $K$ is a path from $1 + \omega$ in $E_2(U)$ then there is a path $K_0$ from $0$ in $U$ such that $K = E_2/K_0$.

**THEOREM 3.2**

If $f$ is an entire function and there exist two points neither of which belongs to the final set of $f$, then $f$ is constant.

**THEOREM 3.3**

If $A$ is a point in the region $R$ and the complement of $R$ contains two points, then there is only one analytic relation $f$ such that

1. $f$ has initial set $R$ and final set the unit-disc $U$,
2. $f^{-1}$ is an analytic function with positive slope at $(0,A)$, and
3. if $K$ is a path from $A$ in $R$, there is a path $K_0$ from $0$ such that $K = f^{-1}/K_0$.

**PROBLEM**

If $A_1$ is a point in the region $R_1$, $A_2$ is a point in the region $R_2$, and each of $R_1$ and $R_2$ has a nondegenerate complement, then there is a reversible analytic function $f$ from $R_1$ onto $R_2$ which has positive slope at $\{A_1,A_2\}$ only in case ..., in which case $f$ consists of all ordered pairs of the form ...
THEOREM 3.4

If \( R \) is a region and \( f \) is a sequence each value of which is an analytic function with initial set \( R \) which does not have either 0 or 1 in its final set, then either there is a subsequence of \( f \) which is continuously convergent on \( R \) or there is a subsequence \( g \) of \( f \) such that \( \frac{1}{g} \) is continuously convergent on \( R \) with limit having only the value 0.

LEMMA 1

If \( 0 < r < 1 \) then there is a number \( b \) in \((0,1)\) such that if \( z \) is a point and \( |z - b| < 1 - b \) then \( |E_2(z) - 1| < r \).

LEMMA 2

If \( 0 < r < 1 \) then there is a number \( b \) in \((0,1)\) such that if \( z \) belongs to \( U \) and to the closure of \( W_1 \) and \( |z - 1| < b \) then \( |z - |z|| < r(1 - |z|) \).

LEMMA 3

There is an analytic function from a simple region onto \( R \).

THEOREM 3.5

If \( f \) is an analytic function from \( E_1(U) \) into \( E_2(U) \) then the point 0 is not an essential singularity of \( f \).

COROLLARY

If the entire function \( g \) is not constant and is not a polynomial then there is a point \( P \) such that each point different from \( P \) is the second term of each of infinitely many ordered pairs in \( g \).

THEOREM 3.6 (Uniformization Theorem)

If \( R \) is a region then there exist a simple region \( D \) and an analytic function \( g \) from \( D \) onto \( R \) having the following properties:

1. \( g^{-1} \) is an analytic relation with initial set \( R \) such that no point in \( R \) is a \( g^{-1} \)-boundary-point of any disc lying in \( R \), and

2. if \( F \) is an analytic relation with initial set \( R \) such that no point in \( R \) is an \( F \)-boundary-point of any disc lying in \( R \), then \( F[g] \) includes an analytic function \( h \) such that \( F \) is the set to which \( Q \) belongs only in case there is a point \( z \) in \( D \) such that \( Q \) is the ordered pair \( \{g(z), h(z)\} \).
DEFINITION

A **well-ordering** of the set $V$ is a subset $R$ or $V \prec V$ such that

1. if $\{x,y\}$ is in $V \prec V$ then either $\{x,y\}$ or $\{y,x\}$ is in $R$,
2. if $\{x,y\}$ is in $R$ then $\{y,x\}$ is in $R$ only in case $y = x$,
3. if $\{x,y\}$ is in $R$ and $\{y,z\}$ is in $R$ then $\{x,z\}$ is in $R$, and,
4. if $W$ is a subset of $V$ then there is a member $x$ of $W$ such that if $y$ belongs
to $W$ then $\{x,y\}$ belongs to $R$.

If $R$ is a well-ordering of the set $V$ then

1. an **initial R-segment** of $V$ is a subset $T$ of $V$ such that either $T = V$ or,
   for each $x$ in $T$ and each $y$ in $V-T$, $\{x,y\}$ belongs to $R$.
2. if $S$ is a subset of $V$ then the $R$-**first member** of $S$ is that member $x$ of $S$
such that if $y$ is in $S$ then $\{x,y\}$ belongs to $R$.

**THEOREM 4.1**

Suppose $M$ is a nondegenerate set, and $G$ is a function from the collection of all subsets
of $M$ such that if $K$ is a subset of $M$ then $G(K)$ is a member of $K$. There is a well-ordering
$R$ of $M$ such that

1. $G(M)$ is the $R$-first member of $M$, and
2. if $H$ is an initial $R$-segment of $M$ different from $M$ then $G(M-H)$ is the $R$-first
member of $M-H$.

**SUGGESTION**

The subset $K$ of $M$ is said to be $G$-normal provided there exists a well-ordering $P$ of $K$
such that $G(M)$ is the $P$-first member of $K$ and, if $H$ is an initial $P$-segment of $K$
different from $K$, $G(M-H)$ is the $P$-first member of $K-H$: in this case, the set $K$ is said
to be $G$-normal with respect to $P$. The following lemmas are true:

**Lemma 1.** If the subset $K$ of $M$ is $G$-normal with respect to $P$ and $H$ is a set then the
following two statements are equivalent:

1. $H$ is an initial $P$-segment of $K$ different from $K$.
2. There is a member $y$ of $K$ different from $G(M)$ such that $x$ belongs to $H$ only in
case $\{x,y\}$ belongs to $P$ and $x \neq y$.

**Lemma 2.** If the subset $K$ of $M$ is $G$-normal with respect to $P$ and to $Q$, then $P$ is $Q$.

**Lemma 3.** If the subset $K$ of $M$ is $G$-normal with respect to $P$ and $H$ is an initial
$P$-segment of $K$, $H$ is $G$-normal with respect to the common part of $P$ and $H \prec H$.

**Lemma 4.** Suppose the subset $K$ of $M$ is $G$-normal with respect to $P$, the subset $L$ of $M$ is
$G$-normal with respect to $Q$, and $K$ is a subset of $L$: $K$ is an initial $Q$-segment of $L$ and
P is the common part of Q and K \( \succcurlyeq \) K.

**Lemma 5.** If K and L are G-normal subsets of M then either K lies in L or L lies in K.

**Lemma 6.** Let V be the set to which x belongs only in case x is a member of some G-normal subset of M, and R be the set to which w belongs only in case w is a member of some well-ordering with respect to which some subset of M is G-normal: V is G-normal with respect to R, and V is M.

**DEFINITION**

A *monotonic collection* is a collection C, each member of which is a set, such that if X and Y are members of C then either X is a subset of Y or Y is a subset of X. If C is a collection, each member of which is a set, then \( C^* \) denotes the set to which z belongs only in case z is a member of some set belonging to C -- and is called the *star* of C.

**THEOREM 4.2**

Suppose the collection S of sets has the property that, for each monotonic subcollection C of S, \( C^* \) belongs to S. If X is a member of S then there is a member of S which includes X and which is not included in any other member of S.

**COROLLARY 1**

If f is an analytic function then there is an analytic function which includes f and which is not included in any other analytic function.

**COROLLARY 2**

If the simple region D lies in the region R then there is a simple region which includes D, which lies in R, and which is not included in any other simple region lying in R.

**THEOREM 4.3**

If M is a nondegenerate set then there is a well-ordering R of M which is *most economical* -- in the sense that if H is an initial R-segment of M different from M then there does not exist a reversible function from M onto H.

**PROBLEM**

If R is a region then there is a simple region D which is *dense in R* -- in the sense that D lies in R and each point of R is either a point of D or a limit-point of D.
Let $M_a$ denote the relation to which $\{p,f\}$ belongs only in case $p$ is a complex number and $f$ is an analytic function with initial set $D$ such that either $D$ is the number-plane or $D$ is a disc with center $p$ and $f$ is not a subset of any other analytic function with initial set a disc with center $p$. Let $R_a$ denote the relation to which $\{X,h\}$ belongs only in case

1) there is a member $\{p,f\}$ of $M_a$ and a disc $d$, having center $p$ and lying in the initial set of $f$, such that $X$ is the subset of $M_a$ to which $\{q,g\}$ belongs only in case $q$ is in $d$ and the sum of $f$ and $g$ is a function, and

2) $h$ is the function to which $\{x,z\}$ belongs only in case $x$ is a member of $X$ and $z$ is the first term of $x$.

Problem 1. $R_a$ is an analytic structure for $M_a$ -- in the sense that $R_a$ is a relation with initial set $Q$ such that

1) each member of $Q$ is a subset of $M_a$ and $Q^*$ is $M_a$,

2) if $\{X,h\}$ is in $R_a$ then $h$ is a reversible function from $X$ onto a region, and

3) if each of $\{X,h\}$ and $\{Y,k\}$ is in $R_a$ and $V$ is the common part of $X$ and $Y$ then $k[h^{-1}]$ is analytic in each component of $h(V)$.

DEFINITION

If $M$ is a set and $R$ is an analytic structure for $M$, the member $P$ of $M$ is a limit-point relative to $R$ of the subset $S$ of $M$ provided there is a member $\{X,h\}$ of $R$ such that $P$ is in $X$ and if $D$ is a disc having center $h(P)$ and lying in $h(X)$ then $h^{-1}(D)$ contains a member of $S$ different from $P$. (Implicitly, meaning has now been assigned to the phrases connected relative to $R$, open relative to $R$, continuous relative to $R$, etc.) If each of $R_1$ and $R_2$ is an analytic structure for the set $M$, $R_1$ is analytically equivalent to $R_2$ provided the sum of $R_1$ and $R_2$ is an analytic structure for $M$.

Problem 2. If $f_0$ is an analytic function with initial set $D_0$ and $\emptyset$ is the relation to which $\{p,x\}$ belongs only in case $p$ is in $D_0$ and $x$ is a member $\{p,f\}$ of $M_a$ such that there is an analytic function which has $p$ in its initial set and which is a subset of $f_0$ and of $f$, then $\emptyset$ is a function from $D_0$ into $M_a$ which is continuous relative to $R_a$.

Problem 3. Let $Z$ be the relation to which $\{F,C\}$ belongs only in case there is a member $\{p,f\}$ of $M_a$ such that (i) $F$ is the analytic relation which includes $f$ and which is not included in any other analytic relation, and (ii) $C$ is the component of $M_a$ relative to $R_a$ to which $\{p,f\}$ belongs: $Z$ is a reversible function.
DEFINITION

An analytic surface is an ordered pair \( \{S, R\} \) such that \( S \) is a set, \( R \) is an analytic structure for \( S \), \( S \) is connected relative to \( R \), and if \( P_1 \) and \( P_2 \) are members of \( S \) then there are mutually exclusive subsets \( T_1 \) and \( T_2 \) of \( S \) which are open relative to \( R \) and contain \( P_1 \) and \( P_2 \), respectively.

REMARK

If \( F \) is an analytic relation which is not a subset of any other analytic relation and \( R \) is the subset of \( R_a \) to which \( \{X, h\} \) belongs only in case \( X \) lies in \( Z(F) \) (see Prob. 3), then \( \{Z(F), R\} \) is an analytic surface --- the Riemann surface of \( F \).

Problem 4. If \( S \) is a region (in the number-plane) and \( R \) is the relation to which \( \{X, h\} \) belongs only in case \( X \) is \( S \) and \( h \) is the contraction of the identity function \( I \) to \( S \), then \( \{S, R\} \) is an analytic surface.

Problem 5. If \( \{S, R\} \) is an analytic surface and \( x_0 \) and \( x_1 \) belong to \( S \), there exist a positive integer \( n \) and a sequence \( \{V_p\} \) of \( \{X, h\} \), each value of which is in the initial set of \( R \), such that \( x_1 \) is in \( V_0 \) and \( x_1 \) is in \( V_n \) and, for each integer \( p \) such that \( 1 \leq p \leq n \), there is a member of \( S \) which belongs to \( V_{p-1} \) and to \( V_p \).

Problem 6. The ordered pair \( \{S, R\} \), such that \( S \) is the number-sphere and \( R \) is the relation to which \( \{X, h\} \) belongs only in case either

1. \( X \) is the set of all members of \( S \) different from \( \{0, 0, 1\} \) and if \( \{a, b, c\} \) is in \( X \) then \( h(a, b, c) = \frac{a+ib}{1-c} \), or
2. \( X \) is the set of all members of \( S \) different from \( \{0, 0, -1\} \) and if \( \{a, b, c\} \) is in \( X \) then \( h(a, b, c) = \frac{a-ib}{1+c} \),

is an analytic surface --- the Riemann sphere --- and \( S \) is compact relative to \( R \) in the sense that each infinite subset of \( S \) has a limit-point relative to \( R \).

Problem 7. If \( \{S, R_1\} \) is an analytic surface, there is an analytic surface \( \{S, R_2\} \) such that \( R_2 \) is analytically equivalent to \( R_1 \) and if \( \{Y, k\} \) belongs to \( R_2 \) then \( k(Y) \) is simple.

DEFINITION

If each of \( \{S_1, R_1\} \) and \( \{S_2, R_2\} \) is an analytic surface, the function \( G \) from \( S_1 \) into \( S_2 \) is analytic relative to \( \{R_1, R_2\} \) provided that if \( \{z, w\} \) is in \( G \) and \( \{X, h\} \) is in \( R_1 \) and \( \{Y, k\} \) is in \( R_2 \) and \( z \) is in \( X \) and \( w \) is in \( Y \) then \( k[G[h^{-1}]] \) is analytic in each component of its initial set.
Problem 8. Reformulate Theorems 25, 62, and 3.2 as statements about analytic surfaces.

Problem 9. Supposing that $F$ is a maximal analytic relation and $\{Z(F), R\}$ is the Riemann surface of $F$, consider the relation $G$ to which $\{z, w\}$ belongs only in case $z$ is a member of $\{p, q\}$ of $Z(F)$ such that $(p, w)$ belongs to $f$.

Problem 10. Supposing that $\{S, R\}$ is an analytic surface and $A$ is a member of $S$, consider the ordered pair $\{S, R\}$ and the function $\emptyset$:

1. $\tilde{S}$ is the relation to which $\{B, C\}$ belongs only in case $B$ is a member of $S$ and $C$ is a set of which there is a member $x$ such that
   
   (i) $x$ is a function from $[0, 1]$ into $S$, continuous relative to $R$, such that $x(0) = A$ and $x(1) = B$ , and
   
   (ii) $y$ belongs to $C$ only in case there is a function $F$ from $[0, 1] \times [0, 1]$ into $S$, continuous relative to $R$, such that $F[I, 0]$ is $x$ and $F[I, 1]$ is $y$ and, for each $t$ in $[0, 1]$, $F(0, t)$ is $A$ and $F(1, t)$ is $B$.

2. $\tilde{R}$ is the relation to which $\{Y, k\}$ belongs only in case there is a member $\{X, h\}$ of $R$, a simple region $D$ lying in $h(X)$, and a member $\{B, C\}$ of $\tilde{S}$ such that $B$ is in $h^{-1}(D)$ and
   
   (i) $Y$ is the subset of $\tilde{S}$ to which $\{B_1, C_1\}$ belongs only in case there is a member $z$ of $C$ and a function $z_1$ from $[0, 1]$ into $h^{-1}(D)$, continuous relative to $R$, such that $z_1(0)$ is $B$ and $z_1(1)$ is $B_1$ and $C_1$ contains the function $x_1$ from $[0, 1]$ such that if $t$ is in $[0, 1/2]$ then $x_1(t)$ is $z(2t)$ but if $t$ is in $[1/2, 1]$ then $x_1(t)$ is $z_1(2t-1)$, and
   
   (ii) $k$ is a function with initial set $Y$ such that if $\{B_1, C_1\}$ is in $Y$ then $h(B_1)$ is the second term of the ordered pair in $k$ of which the first term is $\{B_1, C_1\}$.

3. $\emptyset$ is the function to which $\{u, v\}$ belongs only in case $u$ belongs to $\tilde{S}$ and $v$ is the first term of $u$. 
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