Mathematical Modeling

Quantitative vs. Qualitative model of a physical system. Early X-rays were a useful qualitative tool, because:
(1) X-rays incident upon human body are attenuated at rate $\alpha$ to density of tissue
(2) X-rays travel in straight lines
(3) X-rays darken film Cormack '63 and Hounsfeld '73. X-ray proposed computerized tomography with quantitative theory: one can infer 3D structures from 2D projections. This advance possible because
(1) scintillation crystals to use as detectors
(2) digital computers
Computerized Tomography

We will use X-ray CT as our model problem; many other modalities are possible: MRI, PET, ultrasound, optical electrical impedance. Math is similar for all of these. Math Modeling: numerical parameters that describe the state are called state variables. Model describes the relationship among state variables. State of the system can often be inferred from feasible measurements. May have a linear model $Ax = y$ or a nonlinear model $F(x) = y$; finitely many degrees of freedom (matrix) or infinitely many degrees of freedom (linear operator on infinite dimensional spaces).
Beer’s Law and X-ray Intensity

\[ I(x) = \| I(x) \| \]

\[ \langle I(x), n(x) \rangle |dS| \]

\[ \int \langle I(x), n(x) \rangle dA_S \]

(1) No Refraction or Diffraction: X-ray beams travel along straight lines, not bent by the objects they pass through.

(2) X-rays used are monochromatic: The waves forming the X-ray beam are all of the same frequency.

(3) Beer’s Law: Each material encountered has a linear attenuation coefficient \( \mu \) for x-rays of a given energy. The intensity \( I \) of the x-ray beam satisfies:

\[ \frac{dI}{ds} = \mu(x)I \]
\[ I(s + \delta x) - I(s) \approx -\mu(x)I(s)\delta s \]
Physical Description

\[ i(s) = I(x_0 + sv) \] gives intensity at points along line and \[ m(s) = \mu(x_0 + sv) \]
gives attenuation coefficient.

\[ \frac{di}{ds} = -m(s)i(s) \quad \text{or} \quad \frac{d\log i}{ds} = -m(s) \]
Point Source Device

Attenuation of X-rays along line through the source at angle $\phi$ is given by

$$\frac{dI}{dr} = - \left( \mu_a(r, \phi) + \frac{1}{r} \right) I$$

Coefficient is an effective attenuation coefficient due to attenuation by object and beam spreading.

$$\log \frac{I(r_\phi, \phi)}{I_0(r_0, \phi)} = \log \frac{r_0}{r_\phi} - \int_{a_\phi}^{b_\phi} \frac{b_\phi \mu_a(s, \phi)}{a_\phi} ds$$

Trigonometry gives

$$a_\phi = \ell / \cos \phi \quad b_\phi = L / \cos \phi \quad r_\phi = (L + h) / \cos \phi$$
Suppose $(t, \omega) \in \mathbb{R} \times S^1$ and $\ell_{t,\omega} = \{x \in \mathbb{R} : \langle x, \omega \rangle = t\}$ The Radon transform is defined by

$$\mathcal{R}f(t, \omega) = \int_{\ell_{t,\omega}} f \, ds = \int_{-\infty}^{\infty} f(t\omega + s\omega^\perp) \, ds$$

For $\mathcal{R}$ to be well-defined, $f$ need not be continuous or of bounded support. Sufficient that $f$ be locally integrable and $\int_{-\infty}^{\infty} |f(t\omega + s\omega^\perp)| \, ds < \infty$ for all $(t, \omega) \in \mathbb{R} \times S^1$. “Natural Domain” of the Radon transform:

1. $f$ is regular enough that restriction to a line is locally integrable
2. $f$ decays fast enough that improper integrals converge.
Radon Transform Examples

Neither \( f(x, y) = 1 \) nor \( f(x, y) = \frac{1}{x^2 + y^2} \) are in the natural domain of \( \mathcal{R} \). Why?

Exercise: Show that \( \mathcal{R} \) is Linear: \( \mathcal{R}(\alpha f) = \alpha \mathcal{R}(f) \quad \mathcal{R}(f + g) = \mathcal{R}(f) + \mathcal{R}(g) \)

Even: \( \mathcal{R}(-t, -\omega) = \mathcal{R}f(t, \omega) \)

Monotone: If \( f \) is nonnegative, \( \mathcal{R}f(t, \omega) \geq 0 \) for all \( (t, \omega) \).

Example: Let \( B_1 = B_1(0) \) be the unit disk with center 0. Show that

\[ \mathcal{R}_{\xi B_1}(t, \omega) = \]
Note that $|t| > 1$ implies that $\ell_{t,\omega}$ does not intersect $B_1$.

A function $f$ on $\mathbb{R}^n$ is radial if its value depends only on distance to the origin: $f(x) = F(\|x\|)$ where $F$ is a function of a single variable.

$$\mathcal{R} f(t, \omega) = \int_{-\infty}^{\infty} f(t, s) \, ds = \int_{-\infty}^{\infty} F(\sqrt{t^2 + s^2}) \, ds =$$

Using the change of variable $r^2 = t^2 + s^2$, $2r \, dr = 2s \, ds$, gives

$$\mathcal{R} f(t, \omega) = 2 \int_{t}^{\infty} \frac{F(r)r \, dr}{\sqrt{r^2 - t^2}}$$
The Abel Transform

For $0 < \alpha \leq 1$, the $\alpha$-Abel transform of $g$ is defined by

$$A_\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{g(s) \, ds}{(s - t)^{1-\alpha}}$$

$$A_{\frac{1}{2}}^{-1} = -\partial_t[A_{\frac{1}{2}}]$$

$$\mathcal{R}f(t) = \sqrt{\pi}(A_{\frac{1}{2}}F)(t^2)$$

$$F(r) = -\frac{1}{\pi r} \partial_r \left[ \int_r^\infty \frac{\mathcal{R}f(t) \, t \, dt}{(t^2 - r^2)^{\frac{1}{2}}} \right]$$
\[ A_{\alpha}^{-1} = -\partial_x A_{1-\alpha} \]

\[ \int_x^s \frac{dt}{(t-x)^\alpha(s-t)^{1-\alpha}} = \Gamma(\alpha)\Gamma(1-\alpha) \]

\[ g(t) = A_{\alpha}f = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{f(s)ds}{(s-t)^{1-\alpha}} \]

\[ I = -\partial_x A_{1-\alpha} \circ A_{\alpha} \]

\[ \int_{-\infty}^\infty f(x)g'(x) \, dx = -\int_{-\infty}^\infty f(x)g(x) \, dx \]
Volterra Equations of the First Kind

\[ K f(x) = \int_0^x k(x, y) f(y) \, dy \]

\[ g = f + K f = (I + K) f \]

\[ f = (I + K)^{-1} g \]

Use the Neumann series (obtained from the Taylor expansion for the function \((1 + x)^{-1}\) about \(x = 0\)
\[(I + K)^{-1} f = \sum_{j=0}^{\infty} (-1)^j K^j f\]

\[f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}(\xi) e^{i\langle x, \xi \rangle} \, d\xi\]

We have

\[f(x) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \sigma \hat{f}(\sigma \theta^\perp) e^{i\langle x, \sigma \theta^\perp \rangle} \, d\xi\]

Hence

\[f(x) = \frac{1}{4\pi} \int_{0}^{2\pi} \int_{-\infty}^{\infty} |\sigma| \hat{f}(\sigma \theta^\perp) e^{i\langle x, \sigma \theta^\perp \rangle} \, d\xi\]

\[f(x) = \frac{1}{2} (2\pi)^{-3/2} \int_{0}^{2\pi} \int_{-\infty}^{\infty} |\sigma| P_{\theta} f(\sigma \theta^\perp) e^{i\langle x, \sigma \theta^\perp \rangle} \, d\xi\]
\[ f(x) = \frac{1}{2} (2p)^{-3/2} \int_0^{2\pi} \int_{\mathbb{R}} (\Lambda P_\theta f)(\sigma) e^{i\sigma \langle x, \theta_\perp \rangle} \, d\sigma \, d\phi \]