Lecture 9: The Radon Transform
Radon Transform

Suppose \( f \) is a function from the plane into the reals. For \( (t, \omega) \in \mathbb{R} \times S^1 \) and \( \ell_{t, \omega} = \{ x \in \mathbb{R} : \langle x, \omega \rangle = t \} \) the line determined by \( t \) and \( \omega \), define

\[
\mathcal{R} f(t, \omega) = \int_{\ell_{t, \omega}} f \, ds = \int_{-\infty}^{\infty} f(t\omega + s\omega^{\perp}) \, ds
\]

This is the Radon transform of \( f \) at \( (t, \omega) \), and defines an operator with takes \( f \) and gives back a function defined on ordered pairs \( (t, \omega) \). It is a linear operator on an infinite dimensional space of functions. For \( \mathcal{R} \) to be well-defined, \( f \) need not be continuous or of bounded support. Sufficient that \( f \) be locally integrable and \( \int_{-\infty}^{\infty} |f(t\omega + s\omega^{\perp})| \, ds < \infty \) for all \( (t, \omega) \in \mathbb{R} \times S^1 \). “Natural Domain” of the Radon transform:

1. \( f \) is regular enough that restriction to a line is locally integrable
2. \( f \) decays fast enough that improper integrals converge.
Radon Transform Properties

Neither of the functions $f(x, y) = 1$ nor $f(x, y) = \frac{1}{x^2 + y^2}$ are in the natural domain of $\mathcal{R}$. The constant one function is clearly not integrable over any line in the plane - the integral is not finite. The second function does decay rapidly at $\|x\| \to \infty$, but “blows up” in a neighborhood of the origin. Remember $\mathcal{R}$ is a linear operator mapping one infinite dimensional space into another. Since it is linear, our intuition from matrices in linear algebra carries over in large measure.

1. Linear: $\mathcal{R}(\alpha f) = \alpha \mathcal{R}(f) \quad \mathcal{R}(f + g) = \mathcal{R}(f) + \mathcal{R}(g)$

2. Even: $\mathcal{R}(-t, -\omega) = \mathcal{R}f(t, \omega)$

3. Monotone: If $f$ is nonnegative, $\mathcal{R}f(t, \omega) \geq 0$ for all $(t, \omega)$. 
Closed Form Expressions

Suppose that $\chi_E$ is the characteristic function of the point set $E$ in the plane. The Radon transform of $\chi_E$, is given by $R_{\chi_E}(t, \omega) =$ the length of the intersection of $\ell_{t, \omega} \cap E$. For a concrete example, let $B_1 = B_1(0)$ be the unit disk with center 0. The closed form expression for the transform is

$$R_{\chi_{B_1}}(t, \omega) = \begin{cases} 2\sqrt{1-t^2} & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

Note that $|t| > 1$ implies that $\ell_{t, \omega}$ does not intersect $B_1$. 
Surface Plots Characteristic Function
Intensity Plots Characteristic Function
Radon on Radial

A function $f$ on $\mathbb{R}^n$ is radial if its value depends only on distance to the origin: $f(x) = F(\|x\|)$ where $F$ is a function of a single variable.

$$\mathcal{R} f(t, \omega) = \int_{-\infty}^{\infty} f(t, s) \, ds = \int_{-\infty}^{\infty} F(\sqrt{t^2 + s^2}) \, ds =$$

Using the change of variable $r^2 = t^2 + s^2$, $2r \, dr = 2s \, ds$, gives

$$\mathcal{R} f(t, \omega) = 2 \int_{t}^{\infty} \frac{F(r)r \, dr}{\sqrt{r^2 - t^2}}$$

We want to find $f$, given $\mathcal{R} f$. Hope there is an inverse $\mathcal{R}^{-1}$ from functions on $\mathbb{R} \times S^1$ such that

$$\mathcal{R}^{-1} \circ \mathcal{R} = f$$
Measure Zero

A subset $E \subset \mathbb{R}^n$ has $n$-dimensional measure zero if for any $\varepsilon > 0$ there is a collection of balls $B_{r_i}(x_i)$ so that

$$E \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i)$$

For example any finite point set in $\mathbb{R}^n$ has measure zero, any countable point set has measure zero (for example the set of rationals in $\mathbb{R}^1$ or the ordered rational pairs in $\mathbb{R}^2$ has measure zero. If $f$ is a function on $\mathbb{R}^n$ and the set of points where $f \neq 0$ has measure zero, then

$$\int_{\mathbb{R}^n} |f(\mathbf{x})| \, d\mathbf{x} = 0$$
Measure Zero

If \( \phi \in L^1(\mathbb{R}^n) \), then

\[
\int_{\mathbb{R}^n} f(x) \phi(x) \, dx = 0
\]

This sort of expression is what we realistically measure in image and signal processing. It says that we cannot distinguish a function supported on a set of measure zero from the zero function. We can identify two functions which differ only on a set of measure zero, lumping them into the same equivalence class. This is the approach you took in your measure theory class for the development of the Lebesgue integral.

**Proposition** If \( f \) is a function in the plane, which is supported on a set of measure zero, then the set of values \( (t, \omega) \in \mathbb{R} \times S^1 \) for which \( \mathcal{R} f(t, \omega) \neq 0 \) is itself a set of measure zero.