Mathematics of Manifolds: Lecture 7
Local vs. Global Properties

Surface or 3-manifold has both local and global properties. Local properties are those observable in a small region of the manifold - in the neighborhood of a point. Global properties those observable when the manifold is considered as a whole. Consider a colony of Flatlanders living on a sphere. Which of the following observations are local, which global?

1. Angle of a triangle are 61.2°, 31.7°, and 89.3°

2. Explorer set out to the east and returned from the west

3. Area of Flatland determined to be finite
Local Geometry and Global Topology

It is in the context of “local geometry” and “global topology” that the adjectives “local” and “global” are usually used.

Flat torus and doughnut surface have same global topology, but different local geometries. Flat torus and plane have the same local geometries, but different global topologies.

3-torus has same local geometry as ordinary 3 dimensional space, but global topology very different.
Definition of a Manifold

At last we arrive at the formal definition of a manifold: A two-dimensional manifold is a space with the local topology of a plane, and a three-dimensional manifold is a space with the local topology of $\mathbb{R}^3$. Note that all n-manifolds have the same local topology, that of n-dimensional Euclidean space $\mathbb{R}^n$, but the local topology of a 2-manifold is different from that of at 3-manifold.

A set of points $M$ is defined to be a manifold if each point of $M$ has an open neighborhood which has a continuous 1-1 map onto an open set of $\mathbb{R}^n$ for some $n$. Simply put, this means that $M$ is locally “like” $\mathbb{R}^n$. The dimension of $M$ is $n$. Note that the map is NOT required to preserve lengths, or angles, or any other geometrical notion.
Charts

The map in definition of manifold associates with a point $P$ of $M$ an $n$-tuple $(x_1(P), ..., x_n(P))$. The numbers $x_1(P), ..., x_n(P)$ are called the coordinates of $P$ under the map. Alternate definition of manifold is a set which can be given $n$ independent coordinates in some neighborhood of any point: these coordinates actually define the required map to $\mathbb{R}^n$.

Convention to write the index of the coordinate as a superscript $x^1(P), x^2(P), ..., x^n(P)$ are the $n$ coordinates of $P$ under the map. Note that neighborhood $U$ does not necessarily include all of $M$ (on the sphere it cannot include the whole sphere. There will be other neighborhoods with their own maps - each point of $M$ must lie in one such neighborhood. The pair consisting of a neighborhood and its map is called a chart, the collection of charts is called an atlas.
Homogeneous vs. Nonhomogeneous Geometries

A *homogeneous* manifold is one whose local geometry is the same at all points; the local geometry of a *nonhomogeneous* manifold varies from point to point.

Sphere is homogeneous manifold, whereas the surface of a doughnut surface is nonhomogeneous. Convex around the outside, but saddle-shaped near the hole. However, flat torus is homogeneous at all points.

Major task in study of manifolds is finding homogeneous geometries for manifolds that do not already have them.
Intuitively, *closed* refers to a manifold which is somehow *finite*, while *open* means *infinite*. For those who already know topology, this meaning is different from a point set being open or closed in the topological sense!

Connection with the concept of *bounded* point set: $S$ is a bounded subset of $\mathbb{R}^3$ for example if there is an $r > 0$ such that if $x \in S$ then

$$\|x\| \equiv \sqrt{x_1^2 + x_2^2 + x_3^2} < r$$

A point set $S$ in $\mathbb{R}^n$ is bounded if there exists a closed ball of sufficiently large radius $r$ such that $S$ is included in the ball.
Closed vs. Open Examples

1. A circle - this is a closed manifold
2. A line - this is open, clearly infinite or unbounded
3. A two-holed doughnut surface - closed, finite or bounded
4. A sphere - closed
5. A plane - open
6. An infinitely long cylinder - open
7. A flat torus - closed